

LONG-TIME ASYMPTOTICS FOR THE TODA SHOCK PROBLEM: NON-OVERLAPPING SPECTRA

IRYNA EGOROVA, JOHANNA MICHOR, AND GERALD TESCHL

ABSTRACT. We derive the long-time asymptotics for the Toda shock problem using the nonlinear steepest descent analysis for oscillatory Riemann–Hilbert factorization problems. We show that the half plane of space/time variables splits into five main regions: The two regions far outside where the solution is close to free backgrounds. The middle region, where the solution can be asymptotically described by a two band solution, and two regions separating them, where the solution is asymptotically given by a slowly modulated two band solution. In particular, the form of this solution in the separating regions verifies a conjecture from Venakides, Deift, and Oba from 1991.

1. INTRODUCTION

The investigation of shock waves in the Toda lattice goes back at least to the numerical works of Holian and Straub [16] and Holian, Flaschka, and McLaughlin [15]. A theoretical investigation was later on done by Venakides, Deift, and Oba [33] employing the Lax–Levermore method. As their main result they showed (in the case of some special symmetric initial conditions) that in a sector $|\frac{n}{t}| < \xi'_{cr}$ the solution can be asymptotically described by a period two solution, while in a sector $|\frac{n}{t}| > \xi_{cr}$ the particles are close to the unperturbed lattice. For the remaining region $\xi'_{cr} < |\frac{n}{t}| < \xi_{cr}$ the solution was conjectured to be asymptotically close to a modulated single-phase quasi-periodic solution but this case was not solved there. Despite some follow-up publications by Bloch and Kodama [2, 3] and Kamvissis [18] this problem remained open. The aim of the present paper is to fill this gap. Our method of choice will be the formulation of the inverse scattering problem as a Riemann–Hilbert problem and application of the nonlinear steepest descent analysis developed by Deift and Zhou [7] based on earlier ideas from Manakov [26] and Its [17]. For more on its history and an overview of this method applied to the Toda lattice in the classical case of constant background we refer to [24] (cf. also [19, 25]) and the references therein. Soon after the introduction of this method Deift, Kamvissis, Kriecherbauer, and Zhou [5] applied it to another steplike situation, the Toda rarefaction problem. However, only the case $t \rightarrow \infty$ with n fixed was considered there. In fact, asymptotics in the (n, t) plane require an extension of the original nonlinear steepest descent analysis based on a suitable chosen g -function as first introduced in Deift, Venakides, and Zhou [6]. Recently this was done for the modified Korteweg–de Vries equation by Kotlyarov and Minakov [21, 22, 28] and for the Korteweg–de Vries equation by two of us jointly with Gladka and Kotlyarov

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[9]. However, all these works have in common that the spectra of the underlying Lax operators overlap and hence the associated Riemann surface is simply connected. While Riemann–Hilbert problems on nontrivial Riemann surfaces have a long tradition, see e.g. the monograph by Rodin [29], the nonlinear steepest descent analysis in such situations was developed only recently by Kamvissis and one of us [20] (see also [25, 27]). It is our main novel feature in the present paper to formulate the problem on a Riemann surface formed by combining both spectra and working on this surface. More precisely, in the most interesting region $\xi_{cr,1} < |\frac{n}{t}| < \xi_{cr,2}$ we will work on a dynamically adapted surface.

To describe our results in more detail we recall that the Toda shock problem consists of studying the long-time asymptotics of solutions of the doubly infinite Toda lattice

$$(1.1) \quad \begin{aligned} \dot{b}(n, t) &= 2(a(n, t)^2 - a(n-1, t)^2), \\ \dot{a}(n, t) &= a(n, t)(b(n+1, t) - b(n, t)), \end{aligned} \quad (n, t) \in \mathbb{Z} \times \mathbb{R},$$

with so called steplike shock initial profile

$$(1.2) \quad \begin{aligned} a(n, 0) &\rightarrow a_1, \quad b(n, 0) \rightarrow b_1, \quad \text{as } n \rightarrow -\infty, \\ a(n, 0) &\rightarrow a_2, \quad b(n, 0) \rightarrow b_2, \quad \text{as } n \rightarrow +\infty, \end{aligned}$$

where the background Jacobi operators with constant coefficients

$$(H_j y)(n) = a_j y(n-1) + b_j y(n) + a_j y(n+1), \quad n \in \mathbb{Z}, \quad j = 1, 2,$$

have spectra $\sigma(H_j)$ with the following mutual location: $\inf \sigma(H_1) < \inf \sigma(H_2)$. These spectra can either overlap or not, and it produces essentially different types of asymptotical behavior of the solution.

For the steplike case in the general situation $\sigma(H_1) \neq \sigma(H_2)$ there are two principal cases, distinguished by the conditions $\inf \sigma(H_1) < \inf \sigma(H_2)$ (the Toda shock problem) and $\inf \sigma(H_1) > \inf \sigma(H_2)$ (the Toda rarefaction problem). As mentioned before, the Toda shock problem was studied partly in [33] for non-overlapping background spectra of equal length, and the Toda rarefaction problem in [5] using the Riemann–Hilbert problem approach for finite n only, as $t \rightarrow \infty$, under the restriction that the spectra are again equal in length, non-overlapping, and that the discrete spectrum is symmetric with respect to 0.

In this study we analyze the asymptotical behavior of the solution of the Toda shock problem in the space-time half-plane $(n, t) \in \mathbb{Z} \times \mathbb{R}_+$ (without loss of generality as the results for $\mathbb{Z} \times \mathbb{R}_-$ follow by a simple reflection) in the case of arbitrary non-overlapping background spectra $\sup \sigma(H_1) < \inf \sigma(H_2)$ and under the assumption that the discrete spectrum is absent. We consider the value $\xi := \frac{n}{t}$ as a *slow variable* and propose the precise form of the solution in a vicinity of the rays $\xi = \text{const}$ as usual. We only compute the leading terms of the long-time asymptotics of the solutions, but in all principal regions of the space-time half-plane, excluding small transition regions. Moreover, we assume that no solitons are present to simplify our exposition. They can easily be added using the techniques developed in [25]. We will also not provide detailed error estimates or study the case of overlapping spectra but defer these to forthcoming papers.

To give a short qualitative description of our result (cf. Figure 1), denote $\sigma(H_j) = I_j$, $j = 1, 2$. First of all we find a decreasing sectionally smooth function $\gamma(\xi)$ of the slow variable ξ such that $\gamma(\xi) \rightarrow \pm\infty$ as $\xi \rightarrow \mp\infty$, and such that there are four points $\xi_{cr,1} < \xi'_{cr,1} < \xi'_{cr,2} < \xi_{cr,2}$ with $\gamma(\xi_{cr,1}) = \sup I_2$, $\gamma(\xi'_{cr,1}) = \inf I_2$,

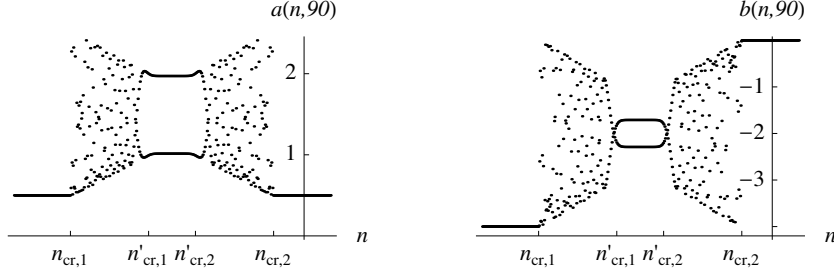


FIGURE 1. The case for a pure step with $\sigma(H_1) = [-5, -3]$ and $\sigma(H_2) = [-1, 1]$.

$\gamma(\xi'_{cr,2}) = \sup I_1$, $\gamma(\xi_{cr,2}) = \inf I_1$. We prove that in the domain $\xi > \xi_{cr,2}$ the solution is asymptotically close to the constant right background solution $\{a_2, b_2\}$, and in the domain $\xi < \xi_{cr,1}$ it is close to the left background $\{a_1, b_1\}$. When the parameter ξ crosses the point $\xi_{cr,2}$ and starts to decay, the point $\gamma(\xi)$ “opens” a band $[\inf I_1, \gamma(\xi)] = I_1(\xi)$. The intervals $I_1(\xi)$ and I_2 can be treated as the bands of a (slowly modulated) finite band solution of the Toda lattice, which turns out to give the leading asymptotical term of our solution with respect to large t . This finite band solution is defined uniquely by its initial divisor. We compute this divisor precisely via the values of the right transmission coefficient on the interval $I_1(\xi)$ (see formulas (5.21), (5.10), (5.22), (5.31), and (5.30) below). Thus, in a vicinity of any ray $\frac{n}{t} = \xi$ the solution of (1.1)–(1.2) is asymptotically finite band (Theorem 5.5). This asymptotical term also can be treated as a function of n , t and $\frac{n}{t}$ in the whole domain $t(\xi'_{cr,2} + \varepsilon) < n < t(\xi_{cr,2} - \varepsilon)$. A numerical comparison between the solution and the corresponding asymptotic formula in this region is shown in Figure 2. When the point $\gamma(\xi)$ crosses the point $\sup I_1$ and until it reaches the point $\inf I_2$, the asymptotic of the solution of (1.1)–(1.2) is one and the same finite band solution, connected with the intervals I_1 and I_2 , whose initial divisor also does not depend on the slow variable ξ . Next, the situation on the domain where γ passes the interval I_2 is the same as when it passed the interval I_1 . The finite band asymptotic here is again local along the ray, and is defined by the intervals I_1 and $I_2(\xi) = [\gamma(\xi), \sup I_2]$.

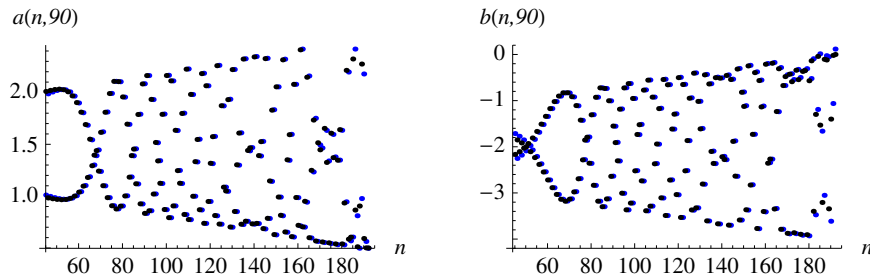


FIGURE 2. Comparison between the solution (black) and the asymptotic formula (blue) in the region $\xi'_{cr,2} < \xi < \xi_{cr,2}$.

We do not study the transition regions in vicinities of the points $\inf I_1$ and $\sup I_2$, but one can expect the appearance of asymptotical solitons here (see [1]).

2. STATEMENT OF THE RIEMANN–HILBERT PROBLEMS

To set the stage we describe the class of initial data which we study. Without loss of generality, by shifting and scaling of the spectral parameter of the Jacobi spectral equation $H(t)\psi = \lambda\psi$, $\lambda \in \mathbb{C}$, we can reduce the asymptotics of the initial data to

$$(2.1) \quad \begin{aligned} a(n, 0) &\rightarrow \frac{1}{2}, & b(n, 0) &\rightarrow 0, & \text{as } n &\rightarrow +\infty, \\ a(n, 0) &\rightarrow d, & b(n, 0) &\rightarrow -c, & \text{as } n &\rightarrow -\infty, \end{aligned}$$

where $c, d \in \mathbb{R}_+$ are constants satisfying the conditions

$$(2.2) \quad c > 1, \quad 0 < 2d < c - 1.$$

We suppose that the initial data decay to their backgrounds exponentially fast

$$(2.3) \quad \sum_{n=-\infty}^0 e^{C|n|} (|a(n, 0) - d| + |b(n, 0) + c|) + \sum_{n=0}^{\infty} e^{Cn} (|a(n, 0) - \frac{1}{2}| + |b(n, 0)|) < \infty,$$

where

$$(2.4) \quad C \geq \max \left\{ c + 2d + |\sqrt{(c + 2d)^2 - 1}|, \frac{1}{2d} \left(1 + c + |\sqrt{(1 + c)^2 - 4d^2}| \right) \right\}.$$

Note that this condition implies that the respective scattering matrix can be continued analytically in a vicinity of the interval $[-c - 2d, 1]$, cf. [13].

Let $a(n, t), b(n, t)$ be the unique solution of the Cauchy problem (1.1) with initial condition of the type (2.1)–(2.3) (cf. [10], [11]). It is known ([12], Lemma 3.2, [32]) that the decay condition (2.3) is preserved by the time evolution of the Toda lattice, and therefore for any fixed t the solution $a(n, t), b(n, t)$ is exponentially small differing from the background constant asymptotics as $n \rightarrow \pm\infty$.

The spectra of the free (background) Jacobi operators,

$$(H_1 y)(n) := dy(n-1) - cy(n) + dy(n+1), \quad (H_2 y)(n) := \frac{1}{2}y(n-1) + \frac{1}{2}y(n+1),$$

are given by $\sigma(H_1) = [-c - 2d, -c + 2d] =: I_1$ and $\sigma(H_2) = [-1, 1] =: I_2$. Under condition (2.2), the spectrum of $H(t)$ consists of an (absolutely) continuous part $[-c - 2d, -c + 2d] \cup [-1, 1]$ of two nonintersecting bands of spectra of multiplicity one, plus a finite number of eigenvalues. We assume that no eigenvalues are present.

In this paper we will use either left or right scattering data, depending on which region we investigate. In order to set up the respective Riemann–Hilbert (RH) problems we need to recall some facts from scattering theory with steplike backgrounds from [11], [12]. We represent the spectral data of the operator $H(t)$ on the upper sheet of the Riemann surface \mathbb{M} connected with the function

$$(2.5) \quad R^{1/2}(\lambda) = -\sqrt{(\lambda^2 - 1)((\lambda + c)^2 - 4d^2)},$$

where $\sqrt{z} = |\sqrt{z}|e^{\frac{i \arg(z)}{2}}$, $-\pi < \arg(z) < \pi$, is the standard root with branch cut along $(-\infty, 0]$. A point on \mathbb{M} is denoted by $p = (\lambda, \pm)$, $\lambda \in \mathbb{C}$, with $p = (\infty, \pm) =$

∞_{\pm} . The projection onto $\mathbb{C} \cup \{\infty\}$ is denoted by $\pi(p) = \lambda$. The sheet exchange map is given by $p^* = (\lambda, \mp)$ for $p = (\lambda, \pm)$. The sets

$$\Pi_U = \{(\lambda, +) \mid \lambda \in \mathbb{C} \setminus (I_1 \cup I_2)\} \subset \mathbb{M}, \quad \Pi_L = \{p^* \mid p \in \Pi_U\},$$

are called upper, lower sheet, respectively. Denote

$$(2.6) \quad \Sigma_{j,u} = \{p = (\lambda + i0, +)\}, \quad \Sigma_{j,\ell} = \{p = (\lambda - i0, +)\}, \quad \lambda \in I_j,$$

and $\Sigma_j = \Sigma_{j,u} \cup \Sigma_{j,\ell}$, $j = 1, 2$. We consider Σ_j as clockwise oriented contours, so $\Sigma_{j,u}$ is passed in positive and $\Sigma_{j,\ell}$ in negative direction. For any function $f(p)$ holomorphic in a neighbourhood of $\Sigma := \Sigma_1 \cup \Sigma_2$ on Π_U and continuous up to the boundary, we consider its value on Σ as

$$f(p) = \lim_{p' \in \Pi_U \rightarrow p} f(p'), \quad p \in \Sigma.$$

We say that f satisfies the symmetry property (SP) at $p \in \Sigma$ if $f(\bar{p}) = \overline{f(p)}$. The points $p = (\lambda + i0, +)$ and $\bar{p} = (\lambda - i0, +)$ are called symmetric points of Σ .

On Π_U , introduce two new spectral variables $z_j(p)$, $j = 1, 2$, by

$$(2.7) \quad z_2(p) = \lambda - \sqrt{\lambda^2 - 1}, \quad z_1(p) = \frac{1}{2d}(\lambda + c - \sqrt{(\lambda + c)^2 - 4d^2}), \quad |z_j(p)| < 1.$$

These variables are different Joukowski transformations of the spectral parameter

$$\lambda = \frac{1}{2} (z_2 + z_2^{-1}) = -c + d (z_1 + z_1^{-1}).$$

The functions $y_2(\lambda, n) = z_2(p)^n$ and $y_1(\lambda, n) = z_1(p)^{-n}$ are the “free exponents” of the background operators H_j . On $\text{clos } \Pi_U := \Pi_U \cup \Sigma$, there exist Jost solutions $\psi_2(p, n, t)$ and $\psi_1(p, n, t)$ of the equation $H(t)\psi(p, n, t) = \lambda\psi(p, n, t)$, $p = (\lambda, +)$, which asymptotically look like the free solutions of the background equations,

$$\lim_{n \rightarrow \infty} z_2^{-n}(p)\psi_2(p, n, t) = 1, \quad \lim_{n \rightarrow -\infty} z_1^n(p)\psi_1(p, n, t) = 1, \quad p \in \text{clos } \Pi_U.$$

We assume that the Wronskian of the Jost solutions

$$W(p, t) := a(n-1, t)(\psi_1(p, n-1, t)\psi_2(p, n, t) - \psi_1(p, n, t)\psi_2(p, n-1, t))$$

does not vanish at the edges of the spectrum, i.e., at the points $-c - 2d, -c + 2d, -1, 1$, which corresponds to a general case without resonances.

The Jost solutions satisfy the scattering relations

$$(2.8) \quad T_2(p, t)\psi_1(p, n, t) = \overline{\psi_2(p, n, t)} + R_2(p, t)\psi_2(p, n, t), \quad p \in \Sigma_2,$$

$$(2.9) \quad T_1(p, t)\psi_2(p, n, t) = \overline{\psi_1(p, n, t)} + R_1(p, t)\psi_1(p, n, t), \quad p \in \Sigma_1,$$

where $T_2(p, t)$, $R_2(p, t)$ (resp. $T_1(p, t)$, $R_1(p, t)$) are the right (resp. left) transmission and reflection coefficients. These coefficients constitute the entries of the scattering matrix. Under condition (2.4) these entries can be continued holomorphically (recall that the discrete spectrum is absent) to the domain $\mathfrak{D} \setminus (I_1 \cup I_2)$, where \mathfrak{D} is the interior of the ellipse

$$(2.10) \quad \mathfrak{D} = \{x + iy \mid x^2(C + 1/C)^{-2} + y^2(C - 1/C)^{-2} < 1\}$$

with C from (2.3), (2.4). We will use this continuation to define $R_j(p)$ on the corresponding region on the upper sheet and extend it to the lower sheet via $R_j(p) = R_j(p^*)$.

Next we collect some known properties of the scattering data which are of relevance for the present paper.

Lemma 2.1 ([12]). *The entries of the scattering matrix satisfy the SP-property on Σ . In addition,*

$$\frac{T_j(p, t)}{T_j(p, t)} = R_j(p, t), \quad p \in \Sigma_j, \quad j = 1, 2.$$

The transmission coefficients are holomorphic on Π_U and continuous up to the boundary except at possibly the edges of the background spectra. They satisfy

$$(2.11) \quad T_2(p, t) = \frac{-\sqrt{\lambda^2 - 1}}{W(p, t)}, \quad T_1(p, t) = \frac{-\sqrt{(\lambda + c)^2 - 4d^2}}{W(p, t)}, \quad p = (\lambda, +),$$

where $W(p, t)$ is the Wronskian of the Jost solutions.

The time evolutions are given by

$$\begin{aligned} R_2(p, t) &= R_2(p) \exp((z_2(p) - z_2^{-1}(p))t), \quad p \in \Sigma_2, \\ R_1(p, t) &= R_1(p) \exp(2d(z_1^{-1}(p) - z_1(p))t), \quad p \in \Sigma_1, \\ T_j(p, t) &= T_j(p) \exp\left(\frac{1}{2}(z_2(p) - z_2^{-1}(p))t - d(z_1(p) - z_1^{-1}(p))t\right), \quad p \in \Pi_U, \end{aligned}$$

where $T_j(p) = T_j(p, 0)$, and $R_j(p) = R_j(p, 0)$, $j = 1, 2$.

In what follows we say that a vector-function $m(p) = (\hat{m}(p), \tilde{m}(p))$ on a Riemann surface \mathbb{M} satisfies the *symmetry condition* if it satisfies

$$(2.12) \quad m(p^*) = m(p) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We say that it satisfies the *normalization condition* if it has limits as $p \rightarrow \infty_{\pm}$ with $\hat{m}(\infty_{\pm})\tilde{m}(\infty_{\pm}) = 1$ and $\hat{m}(\infty_{\pm}) > 0$.

Define vector-valued functions $m_j(p, n, t)$ on Π_U ,

$$(2.13) \quad \begin{aligned} m_2(p, n, t) &= (T_2(p, t)\psi_1(p, n, t)z_2^n(p), \quad \psi_2(p, n, t)z_2^{-n}(p)), \\ m_1(p, n, t) &= (T_1(p, t)\psi_2(p, n, t)z_1^{-n}(p), \quad \psi_1(p, n, t)z_1^n(p)), \end{aligned}$$

and extend them on Π_L by the symmetry condition (2.12). The next lemma justifies that $m_j(p, n, t)$ satisfy the normalization condition too.

Lemma 2.2. *The functions $m_j(p) := m_j(p, n, t)$, $j = 1, 2$, have the following asymptotical behavior as $p \rightarrow \infty_+$*

$$(2.14) \quad m_j(p) = \left(A_j(n) \left(1 - \frac{B_j(n - (-1)^j)}{\lambda}\right), \frac{1}{A_j(n)} \left(1 + \frac{B_j(n)}{\lambda}\right)\right) + O\left(\frac{1}{\lambda^2}\right),$$

where $A_j(n) := A_j(n, t)$, $B_j(n) := B_j(n, t)$ with

$$(2.15) \quad \begin{aligned} A_2(n, t) &= \prod_{j=n}^{\infty} 2a(j, t), & B_2(n, t) &= - \sum_{m=n+1}^{\infty} b(m, t), \\ A_1(n, t) &= \prod_{j=-\infty}^{n-1} \frac{a(j, t)}{d}, & B_1(n, t) &= - \sum_{m=-\infty}^{n-1} (c + b(m, t)). \end{aligned}$$

Proof. Recall (cf. (10.6) in [30]) that the asymptotic expansions of the Jost solutions with respect to large $p = (\lambda, +)$ are given by

$$(2.16) \quad \begin{aligned} \psi_2(p, n, t) &= \frac{1}{A_2(n, t)} \left(\frac{1}{2\lambda} \right)^n \left(1 + \frac{B_2(n, t)}{\lambda} + O(\lambda^{-2}) \right), \\ \psi_1(p, n, t) &= \frac{1}{A_1(n, t)} \left(\frac{\lambda}{d} \right)^n \left(1 + \frac{1}{\lambda} (nc + B_1(n, t)) + O(\lambda^{-2}) \right). \end{aligned}$$

We only give the proof for $m_2(p) = (T_2(p)\psi_1(p, n)z_2^n(p), \psi_2(p, n)z_2^{-n}(p))$ and omit the t -dependence. The behaviour of $\psi_2(\lambda, n)z_2^{-n}$ is evident from (2.16) and

$$z_2 = \lambda - \sqrt{\lambda^2 - 1} = \lambda - \lambda \left(1 - \frac{1}{\lambda^2} \right)^{1/2} = \frac{1}{2\lambda} (1 + O(\lambda^{-2})),$$

For the Wronskian of the Jost solutions (defined after (2.11)) we obtain

$$W(\lambda) = \frac{-a(n)\lambda}{d^{(n+1)}2^n A_1(n+1)A_2(n)} \left(1 + \frac{1}{\lambda} (B_2(n) + (n+1)c + B_1(n+1)) + O(\lambda^{-2}) \right),$$

hence by (2.11) and (2.16),

$$\begin{aligned} T_2(\lambda) &= \frac{d^{n+1}2^n}{a(n)} A_1(n+1)A_2(n) \left(1 - \frac{1}{\lambda} (B_2(n) + (n+1)c + B_1(n+1)) + O(\lambda^{-2}) \right), \\ \psi_1(\lambda, n)z_2^n &= \frac{1}{d^n 2^n A_1(n)} \left(1 + \frac{1}{\lambda} (nc + B_1(n)) + O(\lambda^{-2}) \right), \end{aligned}$$

which yields

$$T_2(\lambda)\psi_1(\lambda, n)z_2^n = A_2(n) \left(1 - \frac{B_2(n-1)}{\lambda} + O(\lambda^{-2}) \right)$$

as desired. \square

We are interested in the jump conditions of $m_j(p)$ on Σ . Denote

$$m_{j,+}(p) = \lim_{\zeta \in \Pi_U \rightarrow p \in \Sigma} m_j(\zeta) \quad \text{and} \quad m_{j,-}(p) = \lim_{\zeta \in \Pi_L \rightarrow p \in \Sigma} m_j(\zeta)$$

in the same point $p \in \Sigma$. One of the steps in applying the nonlinear steepest descent method is to represent the jump conditions for $m_j(p)$ on Σ as matrix-functions of the time variable and the *slow variable* $\xi = \frac{n}{t}$. To this end we first study the phase functions $\Phi_j(p) = \Phi_j(p, \xi)$ given on Π_U by

(2.17)

$$\Phi_1(p) = d(z_1^{-1}(p) - z_1(p)) - \xi \log z_1(p), \quad \Phi_2(p) = \frac{1}{2}(z_2(p) - z_2^{-1}(p)) + \xi \log z_2(p).$$

On Π_L , we define the Joukowski transformations as $z_j(p^*) = z_j^{-1}(p)$, and continue the phase functions as odd functions

$$(2.18) \quad \Phi_j(p^*) = -\Phi_j(p), \quad j = 1, 2.$$

Note that $z_2(p)$ and $z_1(p)$ are not holomorphic on \mathbb{M} , $z_2(p)$ has a jump on Σ_1 and $z_1(p)$ has a jump on Σ_2 . In particular,

$$(2.19) \quad z_1(p) = z_1(\bar{p}) \in \mathbb{R}, \quad z_1(\bar{p}^*) = z_1^{-1}(\bar{p}) = z_1^{-1}(p), \quad p \in \Sigma_2,$$

respectively, $z_2(p)$ is real valued with the same type of jump on Σ_1 . To study the asymptotics in different regions of the space-time plane, we introduce two different Riemann–Hilbert problems connected with the left and right backgrounds.

Recall that to reconstruct a steplike Jacobi operator such as $H(0)$, which has non-overlapping background spectra, it is sufficient to know the value of the reflection coefficient R_j on the upper side of the cut along the spectrum of the corresponding background I_j and the absolute value of the transmission coefficient $|T_j|^2$ on the upper side of the spectrum of the other background.

Denote

$$\chi(p) = - \lim_{p' \in \Pi_U \rightarrow p \in \Sigma} T_1(p', 0) \overline{T_2(p', 0)}, \quad p \in \Sigma.$$

We observe from (2.11) that

$$(2.20) \quad \chi(p) = i|\chi(p)|, \quad p \in \Sigma_u, \quad \chi(p) = -i|\chi(p)|, \quad p \in \Sigma_l,$$

(cf. (2.6)), and therefore $\chi(p) = -\chi(\bar{p})$ for $p \in \Sigma$.

Theorem 2.3. *Suppose that the initial data of the Cauchy problem (1.1)–(2.3) are such that the associated Jacobi operator $H(0)$ does not have a discrete spectrum. Let $\{R_j(p), p \in \Sigma_j, j = 1, 2, \chi(p), p \in \Sigma\}$ be the scattering data of $H(0)$. Then the vector-valued functions $m_j(p) = m_j(p, n, t)$ defined in (2.13), (2.12) are the solutions of the following two vector Riemann–Hilbert problems.*

Find vector-valued functions $m_j(p)$, $j = 1, 2$, which are holomorphic away from the contour Σ on \mathbb{M} and satisfy:

I. The jump condition $m_{j,+}(p, n, t) = m_{j,-}(p, n, t)v_j(p, n, t)$, $j = 1, 2$, where

$$(2.21) \quad v_2(p) = \begin{cases} \begin{pmatrix} 0 & -\overline{R_2(p)}e^{-2t\Phi_2(p)} \\ R_2(p)e^{2t\Phi_2(p)} & 1 \end{pmatrix}, & p \in \Sigma_2, \\ \begin{pmatrix} \chi(p)e^{t(\Phi_{2,+}(p)-\Phi_{2,-}(p))} & e^{-t(\Phi_{2,+}(p)+\Phi_{2,-}(p))} \\ e^{t(\Phi_{2,+}(p)+\Phi_{2,-}(p))} & 0 \end{pmatrix}, & p \in \Sigma_1, \end{cases}$$

$$(2.22) \quad v_1(p) = \begin{cases} \begin{pmatrix} 0 & -\overline{R_1(p)}e^{-2t\Phi_1(p)} \\ R_1(p)e^{2t\Phi_1(p)} & 1 \end{pmatrix}, & p \in \Sigma_1, \\ \begin{pmatrix} \overline{\chi}(p)e^{t(\Phi_{1,+}(p)-\Phi_{1,-}(p))} & e^{-t(\Phi_{1,+}(p)+\Phi_{1,-}(p))} \\ e^{t(\Phi_{1,+}(p)+\Phi_{1,-}(p))} & 0 \end{pmatrix}, & p \in \Sigma_2, \end{cases}$$

II. the symmetry condition

$$(2.23) \quad m_j(p^*) = m_j(p) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad p \in \mathbb{M}, \quad j = 1, 2,$$

III. the normalization condition

$$(2.24) \quad \lim_{p \in \Pi_U \rightarrow \infty_+} m_j(p) = (\hat{m}_j, \tilde{m}_j), \quad \hat{m}_j \cdot \tilde{m}_j = 1, \quad \hat{m}_j > 0.$$

Proof. We only do the proof for m_2 and omit the subscript 2 in the scattering data. Both Jost solutions are holomorphic on Π_U with continuous boundary values for $p \in \Sigma$. Moreover, ψ (resp., ψ_1) is real valued and does not have a jump along Σ_1 (resp., Σ_2). We first derive the jump matrix on Σ_2 . Since z_1 has a jump on Σ_2 (2.19) and $z(p^*) = \bar{z}(\bar{p}^*)$, we symbolically introduce $\psi_1(\bar{p}) = \psi_1(\bar{p}^*) = \psi_1(p) \in \mathbb{R}$, $\psi(\bar{p}) = \bar{\psi}(p)$, and

$$\bar{T} = \bar{T}(p, t) = \overline{T(p)} \exp \left(\frac{1}{2}(\bar{z}(p) - z(p))t + d(z_1^{-1}(p) - z_1(p))t \right).$$

With this notation, $m_-(p) = (\bar{\psi}z^n, \bar{T}\psi_1z^{-n})$ for $p \in \Pi_L$ and $p \in \Sigma_2$. By comparing the ansatz for the jump matrix

$$(T\psi_1z^n, \psi z^{-n}) = (\bar{\psi}z^n, \bar{T}\psi_1z^{-n}) \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

with the scattering relations (2.8) we obtain $\alpha = 0$, $\bar{T}T^{-1}\gamma z^{-2n} = 1$, and $\delta = 1$, $\beta z^{2n} = -\bar{R}$. Hence by Lemma 2.1,

$$\gamma = \frac{T(p)}{\bar{T}(p)} e^{(z-z^{-1})t} z^{2n} = R(p) e^{2t\Phi(z(p))}, \quad \beta = -\frac{\overline{R(p,t)}}{z^{2n}} = -\frac{\overline{R(p)}}{e^{2t\Phi(z(p))}}.$$

To derive the jump matrix on Σ_1 , note that $\Phi(z(\bar{p}^*)) = -\Phi(z(p))$ for $p \in \Sigma_1$. Since $z(p)$ has a jump on Σ_1 , we introduce the notations

$$\begin{aligned} z_+(p) &= \lim_{\zeta \in \Pi_U \rightarrow p \in \Sigma_1} z(p) \in \mathbb{R}, \quad -1 < z_+(p) < 0, \\ z_-(p) &= \lim_{\zeta \in \Pi_L \rightarrow p \in \Sigma_1} z(p) \in \mathbb{R}, \quad z_-(p) < 1, \quad z_-(p) = z_+^{-1}(p). \end{aligned}$$

We denote $\psi(\bar{p}) = \overline{\psi(p)} = \psi(p)$, $\psi_1(\bar{p}) = \overline{\psi_1(p)}$, and

$$\bar{T} = \bar{T}(p, t) = \overline{T(p)} \exp\left(\frac{1}{2}(z_+^{-1}(p) - z_-(p))t + d(z_+^{-1}(p) - z_1(p))t\right).$$

Then

$$(T\psi_1z_+^n, \psi z_+^{-n}) = (\psi z_-^n, \bar{T}\bar{\psi}_1z_-^{-n}) \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

and we get using $z_-(p) = z_+^{-1}(p)$ that $T\psi_1 = \psi\alpha z_-^{2n} + \bar{T}\bar{\psi}_1\gamma$ and $\psi = \psi\beta + \bar{T}\bar{\psi}_1\delta z_-^{2n}$. The left scattering relation (2.9) implies that $\beta = 1$, $\delta = 0$, and $-T = R_1\bar{T}\gamma$ as well as $\alpha = -T_1\bar{T}\gamma z_-^{2n}$. By Lemma 2.1, $\gamma = 1$ and

$$\alpha = -T_1(p)\overline{T(p)}e^{t\frac{1}{2}(z_+(p)-z_+^{-1}(p))-\frac{1}{2}(z_-(p)-z_-^{-1}(p))}z_+^nz_-^{-n} = -T_1(p)\overline{T(p)}e^{\Phi_+-\Phi_-}$$

finishing the proof. \square

We remark that the given solution is the only solution to this Riemann–Hilbert problem. We will prove this in Appendix A.

Our aim is to reduce these RH problems to model problems which can be solved explicitly. To this end we record the following well-known result for easy reference.

Lemma 2.4 (Conjugation). *Let m be the solution of the RH problem $m_+(p) = m_-(p)v(p)$, $p \in \hat{\Sigma}$, on a Riemann surface $\hat{\mathbb{M}}$ which satisfies the symmetry and normalization conditions. Let $\tilde{\Sigma}$ be a contour on $\hat{\mathbb{M}}$ with the same orientation as $\hat{\Sigma}$ on the common part of these contours and suppose that $\hat{\Sigma}$ and $\tilde{\Sigma}$ contain with each point p also p^* . Let D be a matrix of the form*

$$D(p) = \begin{pmatrix} d(p)^{-1} & 0 \\ 0 & d(p) \end{pmatrix},$$

where $d : \hat{\mathbb{M}} \setminus \tilde{\Sigma} \rightarrow \mathbb{C}$ is a sectionally analytic function with $d(p) \neq 0$ except for a finite number of points on $\tilde{\Sigma}$. Set

$$(2.25) \quad \tilde{m}(p) = m(p)D(p),$$

then the jump matrix of the problem $\tilde{m}_+ = \tilde{m}_- \tilde{v}$ is

$$\tilde{v} = \begin{cases} \begin{pmatrix} v_{11} & v_{12}d^2 \\ v_{21}d^{-2} & v_{22} \end{pmatrix}, & p \in \hat{\Sigma} \setminus (\tilde{\Sigma} \cap \hat{\Sigma}), \\ \begin{pmatrix} v_{11}d_+^{-1}d_- & v_{12}d_+d_- \\ v_{21}d_+^{-1}d_-^{-1} & v_{22}d_-^{-1}d_+ \end{pmatrix}, & p \in \tilde{\Sigma} \cap \hat{\Sigma}, \\ \begin{pmatrix} d_+^{-1}d_- & 0 \\ 0 & d_-^{-1}d_+ \end{pmatrix}, & p \in \tilde{\Sigma} \setminus (\tilde{\Sigma} \cap \hat{\Sigma}). \end{cases}$$

If d satisfies $d(p^*) = d(p)^{-1}$ for $p \in \hat{\mathbb{M}} \setminus \tilde{\Sigma}$, then the transformation (2.25) respects the symmetry condition (2.23).

Note that in general, for an oriented contour $\hat{\Sigma}$, the value $f_+(p_0)$ (resp. $f_-(p_0)$) will denote the nontangential limit of the vector function $f(p)$ as $p \rightarrow p_0 \in \hat{\Sigma}$ from the positive (resp. negative) side of $\hat{\Sigma}$, where the positive side is the one which lies to the left as one traverses the contour in the direction of its orientation.

In addition to this Lemma we will apply the technique of so called g -functions in a form proposed in [21]. In contradistinction to [21] we work on the Riemann surface, and these g -functions are in fact Abel integrals on modified Riemann surfaces which are “slightly truncated” with respect to the initial one and depend on the parameter ξ . These Abel integrals approximate the phase functions at infinity up to an additive constant, and transform the jump matrices in a way that allows us to factorize them and to get asymptotically constant matrices on contours. The respective RH problem with constant jump is called the model problem and will be solved explicitly for our case. In the next section we rigorously study the analytical properties of the g -function which approximates the phase Φ_2 in one of the domains, and then list analogous properties of the other g -functions in Section 6.

3. g -FUNCTION: EXISTENCE AND PROPERTIES

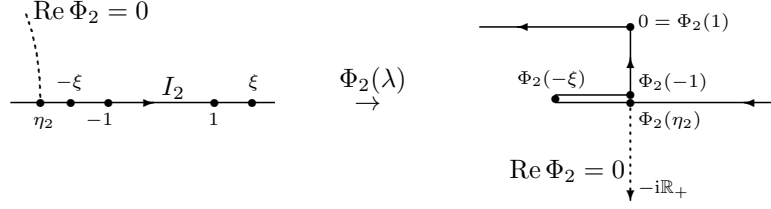
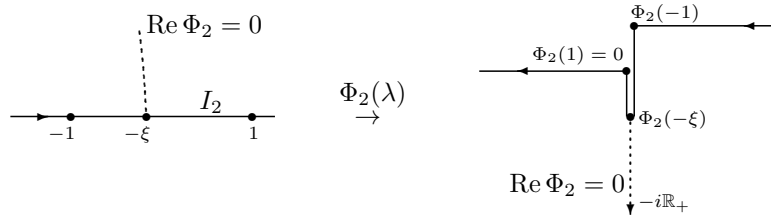
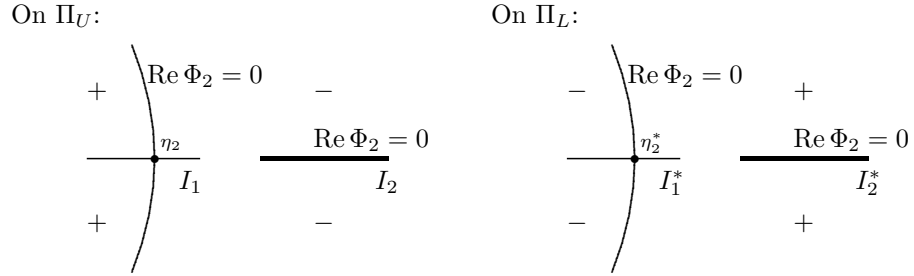
We start with properties of the phase functions which would be desirable to be “inherited” by the g -functions. In accordance with (2.17) and (2.18) we represent $\Phi_j(p)$ for $p = (\lambda, +)$ via the integrals

$$(3.1) \quad \Phi_2(p) = - \int_1^\lambda \frac{x + \xi}{\sqrt{x^2 - 1}} dx, \quad \Phi_1(p) = \int_{-c+2d}^\lambda \frac{x + c + \xi}{\sqrt{(x + c)^2 - 4d^2}} dx,$$

and continue them as odd functions to the lower sheet, $\Phi_j(p^*) = -\Phi_j(p)$, $j = 1, 2$. Then $\Phi_2(p)$ has a jump along the contour Σ_1 and no jump on Σ_2 , respectively, Φ_1 has a jump along Σ_2 and no jump on Σ_1 . For $p = (\lambda, +)$,

$$(3.2) \quad \Phi_j(\lambda \pm i0) = \pm(-1)^j i\pi\xi + \operatorname{Re} \Phi_j(\lambda), \quad \text{for } \lambda \in \begin{cases} (-\infty, -c - 2d], & j = 1, \\ (-\infty, -1], & j = 2, \end{cases}$$

with the natural symmetry on the lower sheet. The jumps of the phase functions along these intervals are equal to $\frac{2\pi i n}{t}$ up to a sign, which implies that $e^{t(\Phi_{j,+}(p) \pm \Phi_{j,-}(p))}$, $j = 1, 2$, which in fact will be used in this paper, have no jumps

FIGURE 3. Conformal map of $\Phi_2(\lambda, \xi)$ for $\xi > 1$ FIGURE 4. Conformal map of $\Phi_2(\lambda, \xi)$ for $\xi \in (-1, 1)$ FIGURE 5. Signature table of $\text{Re } \Phi_2(p)$ for $\eta_2 \in I_1$

along $(-\infty, -c - 2d]$ and $(-\infty, -1]$, respectively. The functions $\Phi_j(p)$ have the following asymptotic behavior as $p \rightarrow \infty_+$

$$(3.3) \quad \begin{aligned} \Phi_2(p, \xi) &= -\lambda - \xi \log \lambda - \xi \log 2 + \frac{1}{2\lambda} + O(\lambda^{-2}), \\ \Phi_1(p, \xi) &= \lambda + \xi \log \lambda + c - \xi \log d + \frac{1}{\lambda}(\xi c - 2d^2) + O(\lambda^{-2}). \end{aligned}$$

Now we describe the points η_j , where the curves $\text{Re } \Phi_j(p, \xi) = 0$, $j = 1, 2$, cross the real axis. Observe that for $\xi > 1$, the function $\Phi_2(\lambda, \xi)$ maps the upper half-plane \mathbb{C}^+ conformally to the domain that lies below the polygon in the right picture of Fig. 3. The line $\text{Re } \Phi_2(\lambda, \xi) = 0$ starts at $\eta_2 < -\xi$ for which

$$(3.4) \quad \Phi_2(\eta_2, \xi) = \Phi_2(-1, \xi).$$

Figure 4 demonstrates that the curve $\text{Re } \Phi_2(\lambda, \xi) = 0$ starts at $\eta_2 = -\xi$ when $\xi \in I_2 = [-1, 1]$. Note that the interval I_2 itself is also part of the curve where $\text{Re } \Phi_2 = 0$. In particular, the signature table on \mathbb{M} for Φ_2 in the case $\eta_2 \in I_1$ is given in Fig. 5. We observe that as the parameter ξ decreases from $+\infty$ to $-\infty$, the

point η_2 increases from $-\infty$ to $+\infty$ and the point η_1 decreases from $+\infty$ to $-\infty$. One can expect from the signature table in Fig. 5 that for $\xi = \frac{n}{t} > \xi_{cr,2}$, where $\xi_{cr,2}$ corresponds to $\eta_2 = -c - 2d$, the asymptotical behavior of the solution of (1.1)–(2.3) will be close to the coefficients of the right initial background operator H_2 . Respectively, if $\xi < \xi_{cr,1}$, where $\xi_{cr,1}$ corresponds to $\eta_1 = 1$, the solution will be close to the coefficients of the left background operator H_1 (see Section 8). According to (3.4), $\xi_{cr,2}$ is the solution of the equation $\Phi_2(-c - 2d, \xi) = \Phi_2(-1, \xi)$. Thus

$$\int_{-c-2d}^{-1} \frac{x + \xi_{cr,2}}{\sqrt{x^2 - 1}} dx = 0,$$

and hence

$$(3.5) \quad \xi_{cr,2} = \frac{\sqrt{(c+2d)^2 - 1}}{\log(c+2d + \sqrt{(c+2d)^2 - 1})}.$$

Here $\sqrt{\cdot} > 0$ is used as an arithmetical value of the square root. The point $\xi_{cr,1}$ is the solution of the equation $\Phi_1(1, \xi) = 0$, that is

$$(3.6) \quad \xi_{cr,1} = -\frac{\sqrt{(1+c)^2 - 4d^2}}{\log\left(\frac{1+c+\sqrt{(1+c)^2 - 4d^2}}{2d}\right)}.$$

We observe that $\xi_{cr,1} < -2d$ and $\xi_{cr,2} > 1$. Now let $\nu_1, \nu_2 \in (-c+2d, -1)$ be two points such that

$$(3.7) \quad \int_{-c+2d}^{-1} \frac{(\lambda - \nu_2)(\lambda + c - 2d)}{R^{1/2}(\lambda)} d\lambda = 0, \quad \int_{-c+2d}^{-1} \frac{(\lambda - \nu_1)(\lambda + 1)}{R^{1/2}(\lambda)} d\lambda = 0,$$

that is,

$$(3.8) \quad \nu_1 = \frac{(c-2d)\mathcal{I}_1 + \mathcal{I}_2}{(c-2d)\mathcal{I}_0 + \mathcal{I}_1}, \quad \nu_2 = \frac{\mathcal{I}_1 + \mathcal{I}_2}{\mathcal{I}_0 + \mathcal{I}_1}, \quad \mathcal{I}_\ell = \int_{-c+2d}^{-1} \frac{\lambda^\ell}{R^{1/2}(\lambda)} d\lambda.$$

The integrals can be explicitly evaluated in terms of Jacobi elliptic functions [4] yielding

$$\begin{aligned} \mathcal{I}_0 &= CK(k), \quad k = \sqrt{\frac{c^2 - (2d+1)^2}{c^2 - (1-2d)^2}}, \quad C = \frac{2}{\sqrt{c^2 - (1-2d)^2}} \\ \mathcal{I}_1 &= C \left(4d \Pi\left(\frac{1-c+2d}{1-c-2d}, k\right) - (c+2d)K(k) \right), \\ \mathcal{I}_2 &= \frac{C}{2} (4dK(k) + (1-c-2d)(1+c-2d)E(k)) - c\mathcal{I}_1. \end{aligned}$$

Next let

$$(3.9) \quad \xi'_{cr,2} = -\nu_2 - 2d, \quad \xi'_{cr,1} = -\nu_1 - c + 1.$$

Since $\nu_1, \nu_2 \in (-c+2d, -1)$, then $|\nu_2 - \nu_1| < -1 + c - 2d$. Therefore,

$$\xi'_{cr,1} < \xi'_{cr,2}.$$

In fact, the following inequalities are valid

$$(3.10) \quad \xi_{cr,1} < \xi'_{cr,1} < \xi'_{cr,2} < \xi_{cr,2},$$

where the parameters ξ_{cr} are uniquely defined by (3.5)–(3.9). These inequalities define three regions with different g -functions. In the region $(\xi'_{cr,2}, \xi_{cr,2})$ (resp. $(\xi_{cr,1}, \xi'_{cr,1})$), a g -function will be a good approximation for the phase function Φ_2

(resp. Φ_1), in the middle region a g -function will approximate both phase functions up to the sign. More precisely, in the right (resp. left) region we study the RH problem associated with the right (resp. left) scattering data and in the middle we study both problems and compare solutions. Inequalities (3.10) can be verified directly, but we get them as a byproduct of existence of such g -functions.

Most of the paper deals with the right scattering data and to shorten notation we will omit the index 2 in the notations of scattering data, g -function, and phase function from here on.

Let $\xi < \xi_{cr,2}$ be a parameter. Consider two functions, $\gamma(\xi) \in I_1$ and $\mu(\xi) \in (\gamma(\xi), -1)$, such that the following two conditions are satisfied:

$$(3.11) \quad c + 2d + \gamma(\xi) + 2\mu(\xi) = -2\xi,$$

and

$$(3.12) \quad \int_{\gamma(\xi)}^{-1} \frac{(\lambda - \mu(\xi))(\lambda - \gamma(\xi))}{R^{1/2}(\lambda, \gamma(\xi))} d\lambda = 0,$$

where

$$(3.13) \quad R^{1/2}(\lambda, \gamma) := -\sqrt{(\lambda^2 - 1)(\lambda + c + 2d)(\lambda - \gamma)}.$$

Evidently, we can always choose two points $\gamma \in I_1$ and $\mu(\gamma) \in (\gamma, -1)$ such that (3.12) holds true. Hence our aim is to show that in a given region $\xi \in (\xi'_{cr,2}, \xi_{cr,2})$, (3.11) can also be satisfied.

Lemma 3.1. *The functions $\gamma(\xi) \in I_1$ and $\mu(\xi) \in (\gamma(\xi), -1)$ satisfying (3.11)–(3.12) exist for $\xi \in (\xi'_{cr,2}, \xi_{cr,2})$. On this interval, $\gamma(\xi)$ is decreasing with $\gamma(\xi_{cr,2}) = -c - 2d$ and $\gamma(\xi'_{cr,2}) = -c + 2d$.*

Proof. First of all note that if $\gamma \in I_1$, then $\mu = \mu(\gamma) \in (\gamma, -1)$ is defined as

$$(3.14) \quad \mu(\gamma) = \int_{\gamma}^{-1} \frac{\lambda(\lambda - \gamma)}{R^{1/2}(\lambda, \gamma)} d\lambda \left(\int_{\gamma}^{-1} \frac{\lambda - \gamma}{R^{1/2}(\lambda, \gamma)} d\lambda \right)^{-1}.$$

Evidently, $\mu(\gamma)$ is a continuous function of γ and by the mean value theorem

$$(3.15) \quad \gamma < \mu(\gamma), \quad \forall \gamma \in (-c - 2d, -c + 2d).$$

Now consider μ as a function of ξ defined via (3.11) and insert it into (3.12). Then

$$F(\gamma, \xi) := - \int_{\gamma}^{-1} \frac{\sqrt{\lambda - \gamma}(\lambda + \xi + \frac{\gamma + c + 2d}{2})}{\sqrt{(\lambda^2 - 1)(\lambda + c + 2d)}} d\lambda$$

satisfies $F(\gamma(\xi), \xi) \equiv 0$. Thus $\gamma(\xi)$ is an implicitly given function and

$$\frac{\partial F}{\partial \gamma} \frac{d\gamma}{d\xi} + \frac{\partial F}{\partial \xi} = 0.$$

Since

$$\frac{\partial F}{\partial \gamma} = -\frac{1}{2} \left(\xi + \frac{3\gamma + c + 2d}{2} \right) \int_{\gamma}^{-1} \frac{d\lambda}{R^{1/2}(\lambda, \gamma)}, \quad \frac{\partial F}{\partial \xi} = \int_{\gamma}^{-1} \frac{\lambda - \gamma}{R^{1/2}(\lambda, \gamma)} d\lambda,$$

and $R^{1/2}(\lambda, \gamma)$ does not change its sign on the interval $(\gamma, -1)$, then

$$(3.16) \quad \frac{d\gamma}{d\xi} = \frac{4\mathcal{K}(\xi)}{2\xi + 3\gamma + c + 2d},$$

where

$$\mathcal{K}(\xi) = \int_{\gamma(\xi)}^{-1} \frac{\lambda - \gamma(\xi)}{R^{1/2}(\lambda, \gamma(\xi))} d\lambda \left(\int_{\gamma(\xi)}^{-1} \frac{d\lambda}{R^{1/2}(\lambda, \gamma(\xi))} \right)^{-1} > 0$$

in the region under consideration. We want to show that

$$(3.17) \quad f(\xi) := 2\xi + 3\gamma(\xi) + c + 2d < 0, \text{ for } \xi \in (\xi'_{cr,2}, \xi_{cr,2}).$$

First, we observe that

$$(3.18) \quad \gamma(\xi_{cr,2}) = -c - 2d.$$

Namely, for any γ , the function $\mu(\gamma)$ is defined by (3.14), which for $\gamma = -c - 2d$ is equal to

$$\mu(-c - 2d) = \int_{-c-2d}^{-1} \frac{\lambda}{\sqrt{\lambda^2 - 1}} d\lambda \left(\int_{-c-2d}^{-1} \frac{d\lambda}{\sqrt{\lambda^2 - 1}} \right)^{-1} = -\xi_{cr,2}$$

by (3.5). On the other hand, for $\mu = -\xi_{cr,2}$, $\gamma = -c - 2d$, and $\xi = \xi_{cr,2}$, (3.11) is also satisfied. Thus (3.18) is true and $f(\xi_{cr,2}) = 2(\xi_{cr,2} - c - 2d)$. Moreover, (3.5) and $c + 2d > 1$ imply

$$(3.19) \quad \xi_{cr,2} \leq \frac{c + 2d + 1}{2}.$$

Since $\frac{c+2d+1}{2} < c + 2d$ by (2.2), then $f(\xi_{cr,2}) < 0$ where f is defined by (3.17). Moreover, the function γ is continuously differentiable with respect to ξ at least in the right vicinity of $\xi_{cr,2}$. In fact, it will be continuously differentiable up to the first point ξ where $f(\xi) = 0$. But if $f(\xi) = 0$ then $3\gamma + c + 2d = -2\xi$. On the other hand, by (3.11) $-2\xi = \gamma + 2\mu + c + 2d$. Thus at ξ_0 where $f(\xi_0) = 0$, one obtains $\mu = \gamma$ which contradicts (3.15). Therefore $\frac{d\gamma}{d\xi} < 0$, starting at the point $\gamma = -c - 2d$ where $\xi = \xi_{cr,2}$ and at least up to the point $\gamma = -c + 2d$ where $\xi = \xi'_{cr,2}$. \square

Lemma 3.2. *For any $\xi \in (\xi'_{cr,2}, \xi_{cr,2})$, there exist points $\nu_1(\xi), h(\xi) \in (\gamma(\xi), -1)$ and $\nu_2(\xi) \in \mathbb{R}$ such that*

$$(3.20) \quad \frac{(\lambda - \mu(\xi))(\lambda - \gamma(\xi))}{R^{1/2}(\lambda, \gamma(\xi))} = \Omega(\lambda, \xi) + \xi\omega(\lambda, \xi),$$

with

$$\Omega(\lambda, \xi) = \frac{(\lambda - \nu_1(\xi))(\lambda - \nu_2(\xi))}{R^{1/2}(\lambda, \gamma(\xi))}, \quad \omega(\lambda, \xi) = \frac{\lambda - h(\xi)}{R^{1/2}(\lambda, \gamma(\xi))},$$

where

$$(3.21) \quad (a) \quad \int_{\gamma(\xi)}^{-1} \Omega(\lambda, \xi) d\lambda = 0; \quad (b) \quad \int_{\gamma(\xi)}^{-1} \omega(\lambda, \xi) d\lambda = 0,$$

and

$$(3.22) \quad \Omega(\lambda, \xi) = -1 + O\left(\frac{1}{\lambda^2}\right), \quad \text{as } \lambda \rightarrow \infty.$$

Moreover, the following formula is valid

$$(3.23) \quad \frac{\partial}{\partial \xi} \frac{(\lambda - \mu(\xi))(\lambda - \gamma(\xi))}{R^{1/2}(\lambda, \gamma(\xi))} = \omega(\lambda, \xi).$$

Proof. Denote

$$(3.24) \quad a := a(\xi) = \int_{\gamma}^{-1} \frac{\lambda^2 d\lambda}{R^{1/2}(\lambda + i0, \gamma)} \left(\int_{\gamma}^{-1} \frac{d\lambda}{R^{1/2}(\lambda + i0, \gamma)} \right)^{-1},$$

$$(3.25) \quad b := b(\xi) = \int_{\gamma}^{-1} \frac{\lambda d\lambda}{R^{1/2}(\lambda + i0, \gamma)} \left(\int_{\gamma}^{-1} \frac{d\lambda}{R^{1/2}(\lambda + i0, \gamma)} \right)^{-1}.$$

From condition (3.21) (b), we get

$$(3.26) \quad h := h(\xi) = b(\xi).$$

Given γ , μ and h , we observe that ν_1 and ν_2 are zeros of the polynomial

$$p(\lambda) = (\lambda - \nu_1)(\lambda - \nu_2) = (\lambda - \gamma)(\lambda - \mu) - \xi(\lambda - h)$$

with real valued coefficients. These zeros cannot be complex conjugated, because (3.20), (3.21) (b), and (3.12) imply

$$\int_{\gamma}^{-1} \frac{p(\lambda)d\lambda}{R^{1/2}(\lambda, \gamma)} = 0,$$

moreover, this formula implies that at least one zero belongs to the interval $(\gamma, -1)$. Condition (3.22) is true due to the asymptotical behavior of the l.h.s. of (3.20) and $\omega(\lambda, \xi)$.

To prove (3.23) we observe that

$$(3.27) \quad -\frac{\partial}{\partial \xi} \frac{(\lambda - \mu(\xi))(\lambda - \gamma(\xi))}{R^{1/2}(\lambda, \gamma(\xi))} + \omega(\lambda, \xi) = \frac{2\mu'(\lambda - \gamma) + \gamma'(\lambda - \mu)}{2R^{1/2}(\lambda, \gamma)} + \frac{\lambda - h}{R^{1/2}(\lambda, \gamma)}.$$

By (3.11), $2\mu' + \gamma' = -2$, therefore (3.27) is in turn equivalent to

$$\frac{2\mu'\gamma + \gamma'\mu + 2h}{R^{1/2}(\lambda, \gamma)} = 0.$$

Hence to prove (3.23) amounts to proving

$$(3.28) \quad 2\mu'(\xi)\gamma(\xi) + \gamma'(\xi)\mu(\xi) = -2h(\xi).$$

From (3.16) we have

$$(3.29) \quad \gamma'(\xi) = \frac{4b(\xi) - 4\gamma(\xi)}{2\xi + 3\gamma(\xi) + c + 2d},$$

where $b(\xi)$ is defined by (3.25), and by (3.26) $b(\xi) = h(\xi)$. On the other hand, (3.11) implies $2\mu' = -2 - \gamma'$ and $\mu = -\xi - \frac{\gamma + c + 2d}{2}$. Substituting this into the l.h.s. of (3.28) and using (3.29) yields

$$\begin{aligned} 2\mu'\gamma + \gamma'\mu &= (-2 - \gamma')\gamma + \gamma' \left(-\xi - \frac{\gamma + c + 2d}{2} \right) \\ &= -\frac{\gamma'}{2} (3\gamma + 2\xi + c + 2d) - 2\gamma = -(2b - 2\gamma) - 2\gamma = -2b = -2h, \end{aligned}$$

which proves (3.28). \square

Now, let $\mathbb{M}(\xi)$ be the Riemann surface of the function (3.13), and denote by $\Pi_U(\xi)$ and $\Pi_L(\xi)$ its upper and lower sheets. Set $I_3(\xi) := [\gamma(\xi), -1]$ and $I_4 := (-\infty, -c - 2d]$, and consider these intervals as contours on $\Pi_U(\xi)$ oriented in positive direction. Let $I_3^*(\xi)$ and I_4^* be the respective contours on $\Pi_L(\xi)$ with negative

direction. Denote $\mathbb{D}(\xi) := \mathbb{M}(\xi) \setminus (I_3(\xi) \cup I_3^*(\xi) \cup I_4 \cup I_4^*)$ and for $p \in \Pi_U(\xi) \cup \mathbb{D}(\xi)$ introduce the function

$$(3.30) \quad g(p) := g(p, \xi) = \int_1^p \frac{(\lambda - \mu(\xi))(\lambda - \gamma(\xi))}{R^{1/2}(\lambda, \gamma(\xi))} d\lambda.$$

Continue it as an odd function on the lower sheet,

$$(3.31) \quad g(p^*) = -g(p).$$

Then g is a single-valued function on $\mathbb{D}(\xi)$. By (3.11), it has the asymptotical behaviour

$$(3.32) \quad \Phi(p, \xi) - g(p, \xi) = K(\xi) - \frac{2k(\xi) - 1}{2p} + O\left(\frac{1}{p^2}\right) \quad \text{as } p \rightarrow \infty_+,$$

where $\Phi(p, \xi) := \Phi_2(p, \xi)$; $k(\xi)$ is the real valued coefficient for the term of order $\frac{1}{p}$ in the expansion of $g(p, \xi)$ with respect to large $p \in \Pi_U(\xi)$ (see (3.3)), and

$$(3.33) \quad K(\xi) = \lim_{\lambda \rightarrow +\infty} \int_1^\lambda \left(\frac{(x - \mu(\xi))\sqrt{x - \gamma(\xi)}}{\sqrt{(x^2 - 1)(x + c + 2d)}} - \frac{x + \xi}{\sqrt{x^2 - 1}} \right) dx$$

is a real constant.

To describe the jumps of $g(p, \xi)$ on $I_3 \cup I_3^* \cup I_4 \cup I_4^*$, denote

$$\int_{-c-2d}^{\gamma(\xi)} \frac{(\lambda - \mu(\xi))(\lambda - \gamma(\xi))}{R^{1/2}(\lambda + i0, \gamma(\xi))} d\lambda = iB(\xi), \quad \int_{-1}^1 \frac{(\lambda - \mu(\xi))(\lambda - \gamma(\xi))}{R^{1/2}(\lambda + i0, \gamma(\xi))} d\lambda = iB'(\xi),$$

where the integration is taken on $\Pi_U(\xi)$. By (3.12), $g(p, \xi) = -g(\bar{p}, \xi) \in i\mathbb{R}$ for $p \in \Sigma(\xi) := \Sigma_1(\xi) \cup \Sigma_2$, where $\Sigma_1(\xi)$ is the contour along the interval $[-c-2d, \gamma(\xi)]$, oriented clockwise. Moreover,

$$(3.34) \quad g_+(p, \xi) - g_-(p, \xi) = -2iB'(\xi), \quad \text{for } p \in I_3(\xi) \cup I_3^*(\xi),$$

$$(3.35) \quad g_+(p, \xi) - g_-(p, \xi) = -2i(B + B')(\xi), \quad \text{for } p \in I_4 \cup I_4^*.$$

Lemma 3.3. *The function $\exp(tg(p, \xi))$ has no jump along $I_4 \cup I_4^*$, that is,*

$$(3.36) \quad \exp(t(g_+(p, \xi) - g_-(p, \xi))) = 1, \quad p \in I_4 \cup I_4^*.$$

Proof. Let \mathcal{C}_ρ be a circle with radius ρ and clockwise orientation enclosing the interval $[-c-2d, 1]$ on the upper sheet. By (3.12),

$$P(\xi) := \oint_{\mathcal{C}_\rho} \frac{(\lambda - \mu(\xi))(\lambda - \gamma(\xi))}{R^{1/2}(\lambda, \gamma(\xi))} d\lambda = 2i(B(\xi) + B'(\xi)).$$

On the other hand,

$$P(\xi) = 2\pi i \operatorname{Res}_\infty \frac{(\lambda - \mu(\xi))(\lambda - \gamma(\xi))}{R^{1/2}(\lambda, \gamma(\xi))} = -2\pi i \xi,$$

due to (3.3) and (3.11), which implied (3.32). Hence

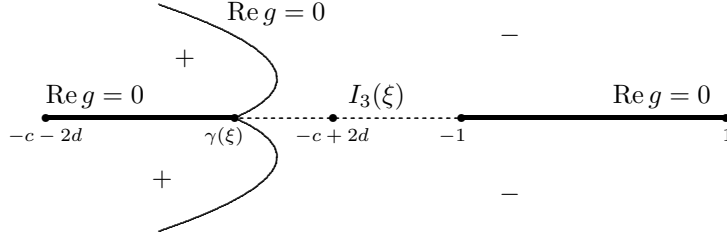
$$(3.37) \quad B(\xi) + B'(\xi) = -\pi \xi,$$

which justifies (3.35). Since $\xi = \frac{n}{t}$, we obtain $tP(\xi) = 2\pi i n$ and thus (3.36). \square

Using this lemma we also replace the jump (3.34) by the jump

$$g_+(p, \xi) - g_-(p, \xi) = 2iB(\xi) + 2\pi i \xi, \quad p \in I_3(\xi) \cup I_3^*(\xi).$$

The signature table for $\operatorname{Re} g$ on the upper sheet of $\mathbb{M}(\xi)$ is depicted in Fig. 6, with opposite signs on the lower sheet due to (3.31).

FIGURE 6. Sign of $\operatorname{Re} g$ on $\Pi_U(\xi)$ 4. REDUCTION TO THE MODEL PROBLEM IN THE DOMAIN $\xi'_{cr,2} < \xi < \xi_{cr,2}$

In this section we perform four basic conjugation-deformation steps which allow us to transform the initial RH problem to an equivalent RH problem with a jump matrix close to a constant matrix for large t , except for neighborhoods of the points $\gamma(\xi)$ and 1. Up to a natural symmetry these steps will be the same in all domains under consideration, and all of them are invertible. Moreover, each step preserves the symmetry condition (2.12) and the normalization condition (2.24).

Consider the solution $m(p) = m_2(p)$ of the “right scattering data” RH problem I, (2.21)–III.

STEP 1. Denote

$$(4.1) \quad d(p) = e^{t(\Phi(p) - g(p))},$$

where $\Phi(p) = \Phi_2(p, \xi)$ and $g(p) = g(p, \xi)$ are defined by (3.1) and (3.30), and set

$$(4.2) \quad m^1(p) = m(p)D(p), \quad D(p) = \begin{pmatrix} d(p)^{-1} & 0 \\ 0 & d(p) \end{pmatrix}.$$

Lemma 2.4 is applicable for this transformation because of (3.31), (2.18). Then m^1 solves the following RH problem on \mathbb{M}

$$(4.3) \quad m_+^1(p) = m_-^1(p)v^1(p),$$

where

$$(4.4) \quad v^1(p) = \begin{cases} \begin{pmatrix} 0 & -\overline{R(p)}e^{-2tg(p)} \\ R(p)e^{2tg(p)} & 1 \end{pmatrix}, & p \in \Sigma_2, \\ \begin{pmatrix} \chi(p) & e^{-2tg(p)} \\ e^{2tg(p)} & 0 \end{pmatrix}, & p \in \Sigma_1(\xi), \\ \begin{pmatrix} \chi(p)e^{t(g_+(p) - g_-(p))} & e^{-t(g_+(p) + g_-(p))} \\ e^{t(g_+(p) + g_-(p))} & 0 \end{pmatrix}, & p \in \tilde{\Sigma}_1(\xi), \\ \begin{pmatrix} e^{t(g_+(p) - g_-(p))} & 0 \\ 0 & e^{-t(g_+(p) - g_-(p))} \end{pmatrix}, & p \in \tilde{I}_3 \cup \tilde{I}_3^*. \end{cases}$$

Here $\Sigma_1(\xi)$ and $\tilde{\Sigma}_1(\xi)$ are contours (oriented clockwise) corresponding to the intervals $[-c-2d, \gamma(\xi)]$ and $[\gamma(\xi), -c+2d]$ on \mathbb{M} , respectively. The intervals $\tilde{I}_3 := [-c+2d, -1]$, $I_4 := (-\infty, -c-2d]$ on Π_U are oriented in positive and $\tilde{I}_3^*, I_4^* \in \Pi_L$ in negative direction. Note that $\tilde{\Sigma}_1(\xi)$ does not belong to the Riemann surface $\mathbb{M}(\xi)$

where the function $g(p)$ is defined, and $g(p)$ has a jump there. In (4.4) we already took Lemma 3.3 into account, which implies

$$\begin{pmatrix} e^{t(g_+(p)-g_-(p))} & 0 \\ 0 & e^{-t(g_+(p)-g_-(p))} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad p \in I_4 \cup I_4^*.$$

The normalization condition (2.24) is now replaced by

$$(4.5) \quad m^1(\infty_+) = m(\infty_+) \begin{pmatrix} e^{-tK(\xi)} & 0 \\ 0 & e^{tK(\xi)} \end{pmatrix}, \quad m^1(\infty_-) = m^1(\infty_+) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which also satisfies (2.24), while the symmetry condition is preserved. Here $K(\xi)$ is defined by (3.33).

STEP 2. Our next aim is to represent the conjugation problem for m^1 as a conjugation problem on the Riemann surface $\mathbb{M}(\xi)$ with jumps along the boundaries of the sheets and along the contours $I_3 = I_3(\xi)$, $I_3^* = I_3^*(\xi)$:

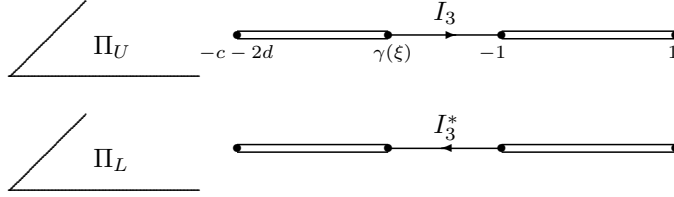


FIGURE 7. Riemann surface $\mathbb{M}(\xi)$ with cuts

Let $m^2(p) = m^1(p)$ for $p \in \mathbb{M}(\xi) \setminus (I_3 \cup I_3^*)$. Since $\tilde{\Sigma}_1(\xi)$ does not belong to $\mathbb{M}(\xi)$, we transform this part of the conjugation problem (4.3) to a conjugation problem on the intervals $I_5(\xi) \subset \Pi_U(\xi)$ and $I_5^*(\xi) \subset \Pi_L(\xi)$ defined by $\pi(I_5(\xi)) = [\gamma(\xi), -c+2d]$. Orientation on these contours preserve the orientation on the contours I_3 and I_3^* , respectively. Let $\tilde{\Sigma}_{1,u}(\xi)$ be the part of $\tilde{\Sigma}_1(\xi)$ which corresponds to $p \in [\gamma(\xi), -c+2d]$ in the sense of (2.6). By the symmetry condition (2.12), the values of $m^2(p)$ as $p \in \Pi_U \rightarrow I_5 - i0$ are the same as

$$(4.6) \quad m^2(p) = m_-^1(p^*) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $m_-^1(p)$ are the former values of m^1 on $\tilde{\Sigma}_{1,u}(\xi)$. Thus on $I_5(\xi)$ we obtain $m_+^2(p) = m_-^2(p)v^2(p)$ with

$$v^2(p) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \chi(p)e^{t(g_+(p)+g_-(p))} & e^{-t(g_+(p)-g_-(p))} \\ e^{-t(g_-(p)-g_+(p))} & 0 \end{pmatrix}, \quad p \in I_5(\xi).$$

Here $g_-(p) = \lim_{p' \in \Pi_U \rightarrow I_5 - i0} g(p')$ and we used (3.31). To simplify this jump matrix recall that the real part of g has no jump on I_5 and that

$$g_{\pm}(p) = \mp iB' + \operatorname{Re} g(p) = \pm iB \mp i\pi\xi + \operatorname{Re} g(p)$$

which follows from (3.37). Taking into account that $e^{2\pi i n} = 1$ we get

$$v^2(p) = \begin{pmatrix} e^{2itB} & 0 \\ \chi(p)e^{2t \operatorname{Re} g(p)} & e^{-2itB} \end{pmatrix}, \quad p \in I_5(\xi).$$

Note that $\operatorname{Re} g(p) < 0$ on I_5 except for the point $\gamma(\xi)$.

On the second half of the contour $\tilde{\Sigma}_1(\xi)$ denoted by $\tilde{\Sigma}_{1,\ell}(\xi)$ (see (2.6)) with orientation from $-c + 2d$ to $\gamma(\xi)$, we use the transformation

$$m_+^2(p) = m_+^1(p^*) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad p \in \Pi_L(\xi), \quad p \rightarrow I_5^*.$$

On the contour I_5^* with orientation from $-c + 2d$ to $\gamma(\xi)$ we have $g_+(p) - g_-(p) = 2iB$. Hence we obtain on I_5^* the conjugation problem $m_+^2(p) = m_-^2(p)v^2(p)$ with

$$v^2(p) = v^1(p) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} e^{2itB} & \chi(p)e^{-2t \operatorname{Re} g(p)} \\ 0 & e^{-2itB} \end{pmatrix}, \quad p \in I_5^*(\xi).$$

Recall that $\operatorname{Re} g(p) > 0$ on I_5^* and we set $\chi(p) = -\chi(p^*)$ for $p \in I_5^*$.

Thus the new RH problem equivalent to the initial problem reads as follows: find a holomorphic function on $\mathbb{M}(\xi) \setminus (I_3 \cup I_3^*)$, which satisfies the jump condition $m_+^2(p) = m_-^2(p)v^2(p)$ with jump matrix

$$v^2(p) = \begin{cases} \begin{pmatrix} 0 & -\overline{R(p)}e^{-2tg(p)} \\ R(p)e^{2tg(p)} & 1 \end{pmatrix}, & p \in \Sigma_2, \\ \begin{pmatrix} \chi(p) & e^{-2tg(p)} \\ e^{2tg(p)} & 0 \end{pmatrix}, & p \in \Sigma_1(\xi), \\ \begin{pmatrix} e^{2itB} & 0 \\ \chi(p)e^{2t \operatorname{Re} g(p)} & e^{-2itB} \end{pmatrix}, & p \in I_5(\xi), \\ \begin{pmatrix} e^{2itB} & \chi(p)e^{-2t \operatorname{Re} g(p)} \\ 0 & e^{-2itB} \end{pmatrix}, & p \in I_5^*(\xi), \\ \begin{pmatrix} e^{2itB} & 0 \\ 0 & e^{-2itB} \end{pmatrix}, & p \in (I_3 \cup I_3^*) \setminus (I_5 \cup I_5^*), \end{cases}$$

and normalization and symmetry conditions II, III.

STEP 3. The jump matrix $v^2(p)$ on Σ_2 can be factorized using Schur complements in the usual way

$$v^2(p) = \begin{pmatrix} 1 & -\overline{R(p)}e^{-2tg(p)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ R(p)e^{2tg(p)} & 1 \end{pmatrix}.$$

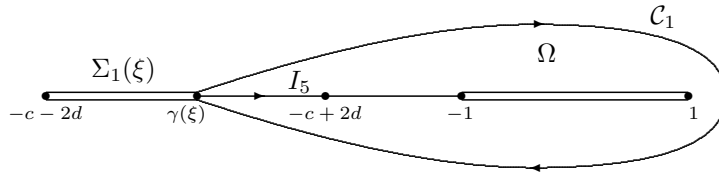


FIGURE 8. Contour deformation for $m^3(p)$ on Π_U

Let Ω be a domain on the upper sheet as in Fig. 8, inside the domain \mathfrak{D} given in (2.10). Recall that due to conditions (2.3), (2.4), the reflection coefficient can be

continued analytically inside \mathfrak{D} , and therefore to Ω . We define $R(p) = R(p^*)$ as $p \in \Omega^*$. This continuation has a jump on $I_5 \cup I_5^*$. Set

$$(4.7) \quad m^3(p) = \begin{cases} m^2(p) \begin{pmatrix} 1 & 0 \\ -R(p)e^{2tg(p)} & 1 \end{pmatrix}, & p \in \Omega, \\ m^2(p) \begin{pmatrix} 1 & -R(p^*)e^{-2tg(p)} \\ 0 & 1 \end{pmatrix}, & p \in \Omega^*, \\ m^2(p), & \text{else.} \end{cases}$$

The jump along Σ_2 disappears. Moreover, on $I_5 \cup I_5^*$ the jump matrix will be

$$v^3(p) = \begin{cases} \begin{pmatrix} e^{2itB} & 0 \\ S(p)e^{2t \operatorname{Re} g(p)} & e^{-2itB} \end{pmatrix}, & p \in I_5(\xi), \\ \begin{pmatrix} e^{2itB} & S(p)e^{-2t \operatorname{Re} g(p)} \\ 0 & e^{-2itB} \end{pmatrix}, & p \in I_5^*(\xi), \end{cases}$$

where $S(p) := R_-(p) - R_+(p) + \chi(p)$. The following lemma can be shown by a straightforward calculation using the Plücker identity as in [9, Lemma 3.2].

Lemma 4.1. *The following identity is valid:*

$$S(p) = R_-(p) - R_+(p) + \chi(p) = 0, \quad p \in I_5 \cup I_5^*.$$

Invoking this lemma we obtain the following RH problem, equivalent to the initial one: find a holomorphic function in $\mathbb{M}(\xi) \setminus (I_3 \cup I_3^*)$, satisfying $m_+^3(p) = m_-^3(p)v^3(p)$, where

$$v^3(p) = \begin{cases} \begin{pmatrix} \chi(p) & e^{-2tg(p)} \\ e^{2tg(p)} & 0 \end{pmatrix}, & p \in \Sigma_1(\xi), \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & p \in \Sigma_2, \\ \begin{pmatrix} e^{2itB} & 0 \\ 0 & e^{-2itB} \end{pmatrix}, & p \in I_3(\xi) \cup I_3^*(\xi), \\ \begin{pmatrix} 1 & 0 \\ -R(p)e^{2tg(p)} & 1 \end{pmatrix}, & p \in \mathcal{C}_1, \\ \begin{pmatrix} 1 & -R(p^*)e^{-2tg(p)} \\ 0 & 1 \end{pmatrix}, & p \in \mathcal{C}_1^*, \end{cases}$$

and standard normalization and symmetry conditions.

STEP 4. Our next conjugation step deals with the factorization of the jump matrix on $\Sigma_1(\xi)$. Consider the following scalar conjugation problem: find a holomorphic function $F(p)$ on $\mathbb{M}(\xi) \setminus (\Sigma_1(\xi) \cup I_3 \cup I_3^*)$ with $F(p^*) = F^{-1}(p)$, bounded at ∞_{\pm} , and satisfying the jump conditions

$$(4.8) \quad F_+(p) = F_-(p) \begin{cases} |\chi(p)|, & p \in \Sigma_1(\xi), \\ e^{i\Delta(\xi)}, & p \in I_3 \cup I_3^*. \end{cases}$$

The value of the real constant $\Delta(\xi)$ is given by (5.10) below. As is shown in Lemma 5.2 below this problem is uniquely solvable and $F(p)$ has a finite real value

at ∞_+ . Given the solution of this problem and taking into account that g has no jump on $\Sigma_1(\xi)$ and $\chi(p) = -i|\chi(p)|$ for $p \in \Sigma_{1,\ell}(\xi)$ by (2.20), we factorize the conjugation matrix $v^3(p)$ on $\Sigma_1(\xi)$ according to

$$v^3(p) = \begin{cases} \begin{pmatrix} F_-^{-1} & 0 \\ \frac{F_-^{-1}}{\chi} e^{2tg(p)} & F_- \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} F_+ & \frac{F_+}{\chi} e^{-2tg(p)} \\ 0 & F_+^{-1} \end{pmatrix}, & p \in \Sigma_{1,u}(\xi), \\ \begin{pmatrix} F_-^{-1} & 0 \\ \frac{F_-^{-1}}{\chi} e^{2tg(p)} & F_- \end{pmatrix} \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} F_+ & \frac{F_+}{\chi} e^{-2tg(p)} \\ 0 & F_+^{-1} \end{pmatrix}, & p \in \Sigma_{1,\ell}(\xi). \end{cases}$$

Introduce "lens" domains Ω_3 and Ω_3^* around the contour $\Sigma_1(\xi)$ as depicted in Fig. 9 with $\Omega_3 \subset \mathfrak{D}$.

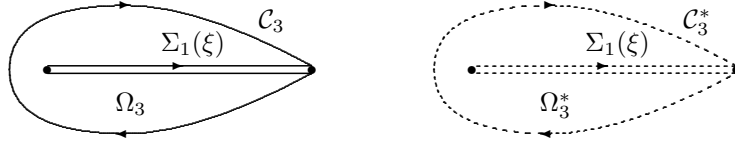


FIGURE 9. The lens contour near $\Sigma_1(\xi)$. Views from the upper and lower sheet. Dotted curves lie in the lower sheet.

We transform the vector $m^3(p)$ as follows:

$$m^4(p) = \begin{cases} m^3(p) \begin{pmatrix} F^{-1}(p) & -\frac{F(p)}{V(p)} e^{-2tg(p)} \\ 0 & F(p) \end{pmatrix}, & p \in \Omega_3, \\ m^3(p) \begin{pmatrix} F^{-1}(p) & 0 \\ \frac{F^{-1}(p)}{V(p)} e^{2tg(p)} & F(p) \end{pmatrix}, & p \in \Omega_3^*, \\ m^3(p) \begin{pmatrix} F^{-1}(p) & 0 \\ 0 & F(p) \end{pmatrix}, & \text{else,} \end{cases}$$

where $V(p)$ is an analytical continuation of $\chi(p) = -\overline{T(p)}T_1(p)$ to the domain Ω_3 and $V(p^*) = -V(p)$ for $p \in \Omega_3^*$. Then m^4 satisfies the standard symmetry and normalization conditions. Using Lemma 4.1 and the results of Lemma 5.2, we

arrive at the following conjugation matrix for $m^4(p)$,

$$v^4(p) = \begin{cases} v^{mod}(p), & p \in \Sigma_1(\xi) \cup I_3(\xi) \cup I_3^*(\xi), \\ \begin{pmatrix} 1 & \frac{F^2(p)}{V(p)}e^{-2tg(p)} \\ 0 & 1 \end{pmatrix}, & p \in \mathcal{C}_3, \\ \begin{pmatrix} 1 & 0 \\ -\frac{F^{-2}(p)}{V(p)}e^{2tg(p)} & 1 \end{pmatrix}, & p \in \mathcal{C}_3^*, \\ \begin{pmatrix} 1 & 0 \\ -F^{-2}(p)R(p)e^{2tg(p)} & 1 \end{pmatrix}, & p \in \mathcal{C}_1, \\ \begin{pmatrix} 1 & -R(p^*)F^2(p)e^{-2tg(p)} \\ 0 & 1 \end{pmatrix}, & p \in \mathcal{C}_1^*, \end{cases}$$

where

$$(4.9) \quad v^{mod}(p) = \begin{cases} \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, & p \in \Sigma_{1,u}(\xi), \\ \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}, & p \in \Sigma_{1,\ell}(\xi), \\ \begin{pmatrix} e^{2itB+i\Delta} & 0 \\ 0 & e^{-2itB-i\Delta} \end{pmatrix}, & p \in I_3(\xi) \cup I_3^*(\xi). \end{cases}$$

Up to now all transformations were invertible. The vector $m^4(p)$ is holomorphic in the domain $\mathbb{M}(\xi) \setminus (\Sigma_1(\xi) \cup I_3(\xi) \cup I_3^*(\xi) \cup \mathcal{C}_i \cup \mathcal{C}_i^*)$. Since the matrices on the contours $\mathcal{C}_i \cup \mathcal{C}_i^*$, $i = 1, 3$, are close to the unit matrix outside of a small neighbourhood of the point $\gamma(\xi)$, we may suppose that the solution of the RH problem for $m^4(p)$ can be approximated by the solution of the following model RH problem: to find a holomorphic vector function

$$(4.10) \quad m^{mod}(p) = (m_1^{mod}(p), m_2^{mod}(p))$$

in $\mathbb{M}(\xi) \setminus (\Sigma_1(\xi) \cup I_3(\xi) \cup I_3^*(\xi))$, bounded at ∞_{\pm} , and satisfying the jump condition

$$(4.11) \quad m_+^{mod}(p) = m_-^{mod}(p)v^{mod}(p)$$

with the jump matrix (4.9), the symmetry condition

$$(4.12) \quad m^{mod}(p^*) = m^{mod}(p) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and the normalization condition

$$(4.13) \quad m_1^{mod}(\infty_+)m_2^{mod}(\infty_+) = 1.$$

5. SOLUTION OF THE MODEL PROBLEM

On the Riemann surface $\mathbb{M}(\xi)$ associated with the function (3.13), introduce a canonical basis of \mathfrak{a} and \mathfrak{b} cycles. The \mathfrak{b} cycle surrounds the interval $[-c - 2d, \gamma(\xi)]$ counterclockwise on the upper sheet $\Pi_U(\xi)$ and the \mathfrak{a} cycle coincides with $I_3 \cup I_3^*$, passing from $\gamma(\xi)$ to -1 on the upper sheet and back from -1 to $\gamma(\xi)$ on the lower

sheet. Let ω_{pp^*} , $p \in \mathbb{M}(\xi) \setminus (I_3 \cup I_3^*)$, be the normalized Abel differential of the third kind with poles at p and p^* such that

$$(5.1) \quad \int_{\mathfrak{a}} \omega_{pp^*} = 0.$$

Then it can be represented as (see [31])

$$(5.2) \quad \omega_{pp^*} = \left(\frac{R^{1/2}(p, \gamma)}{\lambda - \pi(p)} + G(p) \right) \frac{d\lambda}{R^{1/2}(\lambda, \gamma)},$$

where $G(p)$ is a function which implements the normalization condition (5.1). Hence

$$(5.3) \quad G(p) = \frac{R^{1/2}(p, \gamma)}{\Gamma} \int_{\mathfrak{a}} \frac{d\lambda}{(\pi(p) - \lambda) R^{1/2}(\lambda, \gamma)},$$

where we denoted

$$(5.4) \quad \int_{\mathfrak{a}} \frac{d\lambda}{R^{1/2}(\lambda, \gamma)} = \Gamma(\xi) := \Gamma > 0.$$

Lemma 5.1. *Uniformly in λ on $\Sigma(\xi)$ there exists*

$$\lim_{p \rightarrow \infty_{\pm}} \omega_{pp^*} = \omega_{\infty_{\pm} \infty_{\mp}}.$$

Proof. For example, let $p \in \Pi_U$. For $p \rightarrow \infty_+$,

$$(5.5) \quad R^{1/2}(p, \gamma) = -p^2 + \frac{\gamma - c - 2d}{2}p + f(\gamma, p), \quad \text{where } f(\gamma, p) = O(1).$$

Then (5.3) and (5.5) imply

$$\begin{aligned} \Gamma G(p) &= \int_{\mathfrak{a}} \frac{R^{1/2}(p, \gamma)}{p(1 - \frac{\lambda}{p}) R^{1/2}(\lambda, \gamma)} d\lambda = \frac{R^{1/2}(p, \gamma)}{p} \int_{\mathfrak{a}} \frac{(1 + \frac{\lambda}{p} + \frac{\lambda^2}{p^2}) d\lambda}{R^{1/2}(\lambda, \gamma)} + O\left(\frac{1}{p^2}\right) \\ &= \Gamma \frac{R^{1/2}(p, \gamma)}{p} \left(1 + \frac{b}{p} + \frac{a}{p^2}\right) + O\left(\frac{1}{p^2}\right), \end{aligned}$$

where the coefficients $a = a(\xi)$ and $b = b(\xi)$ are defined by (3.24) and (3.25). Substituting this and (5.5) in (5.2) the following holds for any fixed λ as $p \rightarrow \infty_+$

$$\begin{aligned} \omega_{pp^*} &= \frac{R^{1/2}(p, \gamma) + G(p)(\lambda - p)}{(\lambda - p) R^{1/2}(\lambda, \gamma)} = -\frac{\frac{R^{1/2}(p, \gamma)}{p} + (\frac{\lambda}{p} - 1)G(p)}{(1 - \frac{\lambda}{p}) R^{1/2}(\lambda, \gamma)} \\ &= \frac{(1 + \frac{\lambda}{p} + \frac{f_1(\lambda, p)}{p^2})}{R^{1/2}(\lambda, \gamma)} \frac{R^{1/2}(p, \gamma)}{p^2} \left(b - \lambda + \frac{a - \lambda b}{p} + \frac{f_2(\lambda, p)}{p^2}\right) \\ (5.6) \quad &= \frac{\lambda - b}{R^{1/2}(\lambda, \gamma)} + \frac{1}{p} \frac{q(\lambda)}{R^{1/2}(\lambda, \gamma)} + \frac{1}{p^2} \frac{f_3(\lambda, p)}{R^{1/2}(\lambda, \gamma)}, \end{aligned}$$

where

$$(5.7) \quad q(\lambda) := q(\lambda, \xi) = \lambda^2 - \nu(\xi)\lambda + \nu(\xi)b(\xi) - a(\xi), \quad \nu(\xi) = \frac{\gamma(\xi) - c - 2d}{2}.$$

The functions $f_i(\lambda, p, \xi)$, $i = 1, 2, 3$, are uniformly bounded with respect to (λ, p) on any compact set for λ as $p \rightarrow \infty_+$. By (3.21) and (3.26), we see that the first summand in (5.6) corresponds to the definition of $\omega_{\infty_+, \infty_-}$ (see [30]). The case $p \rightarrow \infty_-$ is analogous. \square

Equation (5.3) also implies the continuity of $G(p)$ on $\mathbb{M}(\xi) \setminus (I_3 \cup I_3^*)$. We observe that $G(p)$ has a jump along $I_3 \cup I_3^*$. If this contour is oriented as above, then

$$(5.8) \quad G_+(p) - G_-(p) = -\frac{2\pi i}{\Gamma}, \quad p \in I_3 \cup I_3^*,$$

where the constant Γ is defined by (5.4). To prove (5.8) one applies the Sokhotski–Plemelj formula to (5.3). Respectively, for any $\lambda \in \Sigma_1(\xi) \cup \Sigma_2$,

$$(\omega_{pp^*})_+ - (\omega_{pp^*})_- = -\frac{2\pi i}{\Gamma} \frac{d\lambda}{R^{1/2}(\lambda, \gamma)} = -2\pi i \zeta,$$

where

$$\zeta = \frac{1}{\Gamma} \frac{d\lambda}{R^{1/2}(\lambda, \gamma)}$$

is the holomorphic Abel differential, normalized by the condition $\int_{\mathbf{a}} \zeta = 1$. We proved the following

Lemma 5.2. *The function*

$$(5.9) \quad F(p) := \exp \left(\frac{1}{2\pi i} \int_{\Sigma_1(\xi)} \log |\chi| \omega_{pp^*} \right)$$

solves the conjugation problem (4.8) with

$$(5.10) \quad \Delta(\xi) := i \int_{\Sigma_1(\xi)} \log |\chi| \zeta = \frac{i}{\Gamma} \int_{\Sigma_1(\xi)} \frac{\log |\chi(\lambda)| d\lambda}{R^{1/2}(\lambda, \gamma)} \in \mathbb{R}.$$

Note that this function has finite real limits at ∞_{\pm} by Lemma 5.1. It satisfies $F(p^*) = F^{-1}(p)$ because $\omega_{pp^*} = -\omega_{p^*p}$. Moreover, (5.6) implies that for $p \rightarrow \infty_+$

$$(5.11) \quad F(p) = \exp \left(\frac{1}{2\pi i} \int_{\Sigma_1(\xi)} \log |\chi| \omega_{\infty_+ \infty_-} \right) \left(1 + \frac{Q(\xi)}{p} + O\left(\frac{1}{p^2}\right) \right),$$

where

$$(5.12) \quad Q(\xi) = \int_{\Sigma_1(\xi)} \frac{\log |\chi(\lambda)| q(\lambda, \xi) d\lambda}{2\pi i R^{1/2}(\lambda, \gamma)} \in \mathbb{R},$$

with $q(\lambda, \xi)$ defined by (5.7).

Denote by

$$\tau = \tau(\xi) = \int_{\mathbf{b}} \zeta = -\frac{1}{\Gamma} \int_{\Sigma_1(\xi)} \frac{d\lambda}{R^{1/2}(\lambda, \gamma)}$$

the \mathbf{b} -period of the normalized holomorphic Abel differential ζ . Since $\Gamma > 0$ by (5.4) we have $\tau \in i\mathbb{R}_+$. Introduce the theta function

$$\theta(v) := \theta(v | \tau) = \sum_{m \in \mathbb{Z}} \exp(\pi i m^2 \tau + 2\pi i m v)$$

and the Abel map $A(p) := A(p, \xi) = \int_{-c-2d}^p \zeta$, which has the following properties (cf. [14, 30]):

$$(5.13) \quad \begin{aligned} A(p^*) &= -A(p), & A(\gamma) &= -\frac{\tau}{2} \pmod{\tau}, \\ A_+(p) - A_-(p) &= -\tau \quad \text{for } p \in I_3 \cup I_3^*, \\ A(p) &= A(\infty_{\pm}) \pm \frac{1}{\Gamma\lambda} + O\left(\frac{1}{\lambda^2}\right), \quad p = (\lambda, \pm) \rightarrow \infty_{\pm}, \end{aligned}$$

$$(5.14) \quad 2A(\infty_+) = A(\infty_+) - A(\infty_-) = \frac{1}{2\pi i}\lambda,$$

where

$$(5.15) \quad \lambda = \lambda(\xi) = \int_b \omega_{\infty_+ \infty_-}.$$

Let $\Xi = \frac{\tau}{2} + \frac{1}{2}$ be the Riemann constant. Then the functions $\theta(A(p) + \frac{\tau}{2} - \Xi)$ and $\theta(A(p^*) + \frac{\tau}{2} - \Xi)$ both have their only zeros at γ . Denote

$$(5.16) \quad \alpha(p) := \alpha(p, \xi) = \frac{\theta(A(p) + \frac{\tau(\xi)}{2} - \Xi(\xi) + \frac{tB(\xi)}{\pi} + \frac{\Delta(\xi)}{2\pi})}{\theta(A(p) + \frac{\tau(\xi)}{2} - \Xi(\xi))}.$$

Standard properties of theta functions show that this function is holomorphic on $\mathbb{M}(\xi)$ except at the point γ , and has a single jump on the set $I_3 \cup I_3^*$,

$$\begin{aligned} \alpha_+(p) &= \alpha_-(p) e^{2itB(\xi) + i\Delta(\xi)}, \\ \alpha_+(p^*) &= \alpha_-(p^*) e^{-2itB(\xi) - i\Delta(\xi)}. \end{aligned}$$

Introduce now on the upper sheet $\Pi_U(\xi)$ the function

$$(5.17) \quad \delta(p) = \sqrt[4]{\frac{\pi(p) - \gamma}{\pi(p) + c + 2d}},$$

where the branch of $\sqrt[4]{\cdot}$ is chosen to take positive values for $\pi(p) > \gamma$. Respectively, this function also takes positive values for $p < -c - 2d$. We continue $\delta(p)$ as an even function to $\Pi_L(\xi)$ by $\delta(p^*) = \delta(p)$. Observe that $\delta(p)$ has no jump on Σ_2 , because it is real valued and takes equal values at symmetric points of I_2 on Π_U . On $\Sigma_1(\xi)$, it solves the following conjugation problem:

$$\delta_+(p) = \delta_-(p) \begin{cases} i, & p \in \Sigma_{1,u}(\xi), \\ -i, & p \in \Sigma_{1,\ell}(\xi). \end{cases}$$

Thus the vector $\hat{m}(p) = (\delta(p)\alpha(p), \delta(p^*)\alpha(p^*))$ solves the jump problem $\hat{m}_+(p) = \hat{m}_-(p)v^{mod}(p)$, where $v^{mod}(p)$ is defined by (4.9), and satisfies the symmetry condition (2.12). But $\hat{m}(p)$ does not satisfy the normalization condition, because $\hat{m}_1(\infty_+)\hat{m}_2(\infty_+) \neq 1$. To mend this, recall that $\delta(\infty_+) = \delta(\infty_-) = 1$. Then

$$(5.18) \quad m^{mod}(p) = \frac{1}{\sqrt{\alpha(\infty_+)\alpha(\infty_-)}} (\delta(p)\alpha(p), \delta(p^*)\alpha(p^*))$$

solves the model problem including (4.13). Note that this solution is bounded everywhere in $\mathbb{M}(\xi)$ except at the branch points $-c-2d, \gamma(\xi)$, where both components of $m^{mod}(p)$ have singularities of type $O((p+c+2d)^{-1/4})$ and $O((p-\gamma)^{-1/4})$.

Our next task is to derive the asymptotic formula for this vector as $p \rightarrow \infty_+$ up to a summand of order $O(p^{-2})$.

Lemma 5.3. *Let Ω_0 be the Abel differential of the second kind on $\mathbb{M}(\xi)$ with second order poles at ∞_+ and ∞_- normalized by the condition $\int_a \Omega_0 = 0$ and let $\omega_{\infty_+\infty_-}$ be the Abel differential of the third kind as above. Then*

$$\frac{\partial}{\partial p} g(p, \xi) d\pi = \Omega_0 + \xi \omega_{\infty_+\infty_-},$$

where $\frac{\partial}{\partial p} g(p, \xi)$ is defined by (3.30). Respectively,

$$-2iB(\xi) = U + \xi\lambda,$$

where $U = \int_b \Omega_0$ is the b -period of Ω_0 and λ is the b -period of $\omega_{\infty_+\infty_-}$ defined by (5.15).

This lemma is a direct corollary of Lemma 3.2 and (3.20). Since $\xi = \frac{n}{t}$ we have

$$\frac{tB(\xi)}{\pi} = -t \frac{U}{2\pi i} - n \frac{\lambda}{2\pi i}$$

and substituting this into (5.16), taking into account that $\theta(v+1) = \theta(v)$, we get

$$(5.19) \quad \alpha(p) = \frac{\theta(A(p) + \frac{\tau(\xi)}{2} - \Xi(\xi) - t \frac{U}{2\pi i} - n \frac{\lambda}{2\pi i} + \frac{\Delta(\xi)}{2\pi})}{\theta(A(p) + \frac{\tau(\xi)}{2} - \Xi(\xi))}.$$

Passing to the limit as $p \rightarrow \infty_+$ and using property (5.14) we obtain

$$\alpha(\infty_+) = \frac{\theta(\underline{z}(n-1, t))}{\alpha^+}, \quad \alpha(\infty_-) = \frac{\theta(\underline{z}(n, t))}{\alpha^-},$$

where

$$(5.20) \quad \underline{z}(n, t) := A(\infty_+) - n \frac{\lambda}{2\pi i} - t \frac{U}{2\pi i} + \frac{\tau(\xi)}{2} + \frac{\Delta(\xi)}{2\pi} - \frac{\lambda}{2\pi i} - \Xi(\xi),$$

and

$$\alpha^\pm := \alpha^\pm(\xi) = \theta\left(A(\infty_\pm) + \frac{\tau(\xi)}{2} - \Xi(\xi)\right).$$

Note that since $\frac{\Delta(\xi)}{2\pi} - \frac{\lambda}{2\pi i} \in \mathbb{R}$ and $\frac{\tau(\xi)}{2} = -A(\gamma)$, by the Jacobi inversion theorem ([30]), there exists a point $\rho(\xi) \in I_3 \cup I_3^*$ such that

$$(5.21) \quad -A(\rho) + A(\gamma) = \frac{\Delta(\xi)}{2\pi} - \frac{\lambda}{2\pi i}.$$

Thus we can represent $\underline{z}(n, t)$ in a more familiar form for finite band operators

$$(5.22) \quad \underline{z}(n, t) := A(\infty_+) - A(\rho) - n \frac{\lambda}{2\pi i} - t \frac{U}{2\pi i} - \Xi(\xi).$$

Passing to the limit $p \rightarrow \infty_+$ in the first component of vector (5.18) we get

$$(5.23) \quad m_1^{mod}(\infty_+) = \beta(\xi) \sqrt{\frac{\theta(\underline{z}(n-1, t))}{\theta(\underline{z}(n, t))}},$$

where

$$(5.24) \quad \beta(\xi) = \sqrt{\frac{\alpha^-}{\alpha^+}} = \sqrt{\frac{\theta(A(\infty_-) + \frac{\tau(\xi)}{2} - \Xi(\xi))}{\theta(A(\infty_+) + \frac{\tau(\xi)}{2} - \Xi(\xi))}} \in \mathbb{R}$$

is a real valued continuously differentiable function of ξ with a bounded derivative on the interval of investigation. Apply to (5.23) the formulas (4.5), (3.33), (5.9),

and Lemma 5.1. We obtain that the first component of the solution of the initial RH problem, which is equal to $\prod_{j=n}^{+\infty} (2a(j, t))$ by (2.15) and (2.14), is

$$\prod_{j=n}^{+\infty} (2a(j, t)) = \beta(\xi) e^{tK(\xi)} e^{\frac{1}{2\pi i} \int_{\Sigma_1(\xi)} \log |\chi| \omega_{\infty+} -} \sqrt{\frac{\theta(\underline{z}(n-1, t))}{\theta(\underline{z}(n, t))}} (1 + o(1)),$$

where $\beta(\xi)$ and $\underline{z}(n, t)$ are defined by (5.24) and (5.22), and $o(1)$ is a function which tends to 0 as $t \rightarrow \infty$. Recall that we treat ξ as a slow variable, and that all functions of ξ are continuously differentiable with respect to ξ in our considerations. This means, for example, that if $\xi = \frac{n}{t}$ and $\xi' = \frac{n+1}{t}$, then $\beta(\xi)\beta^{-1}(\xi') = 1 + O(\frac{1}{t})$ and

$$e^{t(K(\xi) - K(\xi'))} = e^{-\frac{dK}{d\xi}(\xi)} + O\left(\frac{1}{t}\right),$$

etc. Consequently,

$$(5.25) \quad a^2(n, t) = \frac{1}{4} e^{-2K'(\xi)} \frac{\theta(\underline{z}(n-1, t))\theta(\underline{z}(n+1, t))}{\theta^2(\underline{z}(n, t))} + o(1).$$

Lemma 5.4. *Let $\tilde{a} = \tilde{a}(\xi)$ be a constant from the expansion (cf. [30, equ. (9.42)])*

$$e^{\int_{c-2d}^p \omega_{\infty+} -} = -\frac{\tilde{a}}{\lambda} \left(1 + \frac{\tilde{b}}{\lambda} + O\left(\frac{1}{\lambda^2}\right) \right), \quad p = (\lambda, +) \in \Pi_U(\xi),$$

and let $K'(\xi) := \frac{dK}{d\xi}(\xi)$, $k'(\xi) = \frac{dk}{d\xi}(\xi)$, where $K(\xi)$ and $k(\xi)$ were defined in (3.33) and (3.32). Then

$$(5.26) \quad \frac{1}{4} e^{-2K'(\xi)} = \tilde{a}^2(\xi),$$

$$(5.27) \quad k'(\xi) = \tilde{b}(\xi).$$

Proof. According to equation (9.43) in [30], the constant \tilde{a} can be computed as

$$\log \tilde{a} = \lim_{\lambda \rightarrow +\infty} \left(\int_1^\lambda \omega_{\infty+} - + \log \lambda \right).$$

Therefore, one has to prove that $K'(\xi) + \log 2 = -\log \tilde{a}$. By (3.33) and (3.23),

$$\begin{aligned} & \frac{d}{d\xi} \lim_{\lambda \rightarrow +\infty} \int_1^\lambda \left(\frac{(x - \mu(\xi))\sqrt{x - \gamma(\xi)}}{\sqrt{(x^2 - 1)(x + c + 2d)}} - \frac{x + \xi}{\sqrt{x^2 - 1}} \right) dx + \log 2 \\ &= - \lim_{\lambda \rightarrow +\infty} \left(\int_1^\lambda \frac{\partial^2}{\partial \xi \partial x} \tilde{g}(x, \xi) dx + \log \left(\lambda + \sqrt{\lambda^2 - 1} \right) \right) + \log 2 \\ &= - \lim_{\lambda \rightarrow +\infty} \left(\int_1^\lambda \omega_{\infty+} - + \log \lambda \right), \end{aligned}$$

which proves (5.26). Formula (5.27) follows immediately from (3.23). \square

To derive the term of order λ^{-1} for the first component of the solution we observe that by (5.13), (5.19), and (5.20),

$$(5.28) \quad \alpha(p) = \alpha(\infty_+) \left(1 + \frac{1}{\Gamma \lambda} \frac{\partial}{\partial w} \log \left(\frac{\theta(\underline{z}(n-1, t) + w)}{\theta(A(\infty_+) + \frac{\tau}{2} - \Xi + w)} \right) \Big|_{w=0} + O\left(\frac{1}{\lambda^2}\right) \right),$$

$p = (\lambda, +)$. Denote

$$(5.29) \quad \eta := \eta(\xi) = -\frac{1}{\Gamma} \frac{\partial}{\partial w} \log \left(\theta \left(A(\infty_+) + \frac{\tau}{2} - \Xi + w \right) \right) \Big|_{w=0} - \frac{\gamma + c + 2d}{4}.$$

Then combining (5.18), (5.23), (5.17), (5.28), and (5.29) we get

$$m_1^{mod}(p) = m_1^{mod}(\infty_+) \left(1 + \frac{1}{\Gamma \lambda} \frac{\partial}{\partial w} \log \theta(\underline{z}(n-1, t) + w) \Big|_{w=0} + \frac{\eta}{\lambda} + O\left(\frac{1}{\lambda^2}\right) \right).$$

Respectively, by (4.1), (4.2), (5.11), (5.12), (5.7), (3.32), and (2.14)

$$-B_2(n-1, t) = \left(-tk(\xi) + \frac{t}{2} + \eta(\xi) + Q(\xi) + \frac{1}{\Gamma} \frac{\partial}{\partial w} \log \theta(\underline{z}(n-1, t) + w) \Big|_{w=0} \right) (1 + o(1)),$$

where $o(1)$ tends to 0 as $t \rightarrow \infty$ and $\frac{n}{t} = \xi$ is almost a constant. Apply now the same arguments as for (5.25) and use (5.27) to get $b(n, t) = b_q(n, t) + o(1)$, where

$$(5.30) \quad b_q(n, t) = \tilde{b} + \frac{1}{\Gamma} \frac{\partial}{\partial w} \log \left(\frac{\theta(\underline{z}(n-1, t) + w)}{\theta(\underline{z}(n, t) + w)} \right) \Big|_{w=0}.$$

Lemma 5.4 and (5.25) also imply $a^2(n, t) = a_q^2(n, t) + o(1)$ with

$$(5.31) \quad a_q^2(n, t) = \tilde{a}^2 \frac{\theta(\underline{z}(n-1, t))\theta(\underline{z}(n+1, t))}{\theta^2(\underline{z}(n, t))}.$$

Formulas (5.31) and (5.30) describe a classical two band Toda lattice motion (cf. [30], Theorem 9.48). Thus, assuming that the solution of the model RH problem (4.9)–(4.13) approximates well the solution of the initial RH problem I, (2.21)–III, we arrive at the following

Theorem 5.5. *Let*

1. $\{a(n, t), b(n, t)\}$ be the solution of the problem (1.1), (2.1)–(2.4) as $n, t \rightarrow \infty$ in the domain $\xi'_{cr,2} t < n < \xi_{cr,2} t$, where the parameters $\xi'_{cr,2}$ and $\xi_{cr,2}$ are defined by (3.9), (3.7), and (3.5);
2. $\xi \in (\xi'_{cr,2}, \xi_{cr,2})$ be a parameter; $\gamma(\xi) \in (-c - 2d, -c + 2d)$ be defined by Lemma 3.1; $\mathbb{M}(\xi)$ be the Riemann surface of the function (3.13);
3. $\rho(\xi)$ be a point on $\mathbb{M}(\xi)$, defined by (5.21), (5.10) with $\pi(\rho) \in [\gamma, -1]$;
4. $\{a_q(n, t), b_q(n, t)\} = \{a_q(n, t, \xi), b_q(n, t, \xi)\}$ be the finite band solution corresponding to the initial divisor $\rho(\xi)$ via (5.31), (5.30), (5.22).

Then in a vicinity of any ray $n = \xi t$ the solution of the problem (1.1), (2.1)–(2.4) has the following asymptotical behavior as $t \rightarrow +\infty$

$$a^2(n, t) = a_q^2(n, t, \xi) + o(1), \quad b(n, t) = b_q(n, t, \xi) + o(1).$$

6. ASYMPTOTICS OF THE SOLUTION IN THE DOMAIN $\xi_{cr,1} < \xi < \xi'_{cr,1}$

In this domain we study the asymptotic behavior of the solution to the problem (1.1)–(2.4) with the help of the RH problem I, (2.22)–III. The considerations are similar to those in Sections 3–5, and we give a short description of the necessary changes. Let $\xi \in (\xi_{cr,1}, \xi'_{cr,1})$, where the points $\xi_{cr,1}$ and $\xi'_{cr,1}$ are defined by (3.6), (3.7), (3.9). We choose the g -function here with its moving point $\gamma_1 = \gamma_1(\xi)$ on the interval $(-1, 1)$:

$$g_1(p, \xi) = - \int_1^p \frac{(x - \mu_1(\xi))(x - \gamma_1(\xi))}{R^{1/2}(x, \gamma_1(\xi))} dx,$$

where the function

$$(6.1) \quad R_1^{1/2}(\lambda, \gamma_1) = -\sqrt{(\lambda + c + 2d)(\lambda + c - 2d)(\lambda - 1)(\lambda - \gamma_1)}$$

defines the respective Riemann surface $M_1(\xi)$. The points $\mu_1(\xi) \in (-c + 2d, \gamma_1(\xi))$ and $\gamma_1(\xi) \in (-1, 1)$ are chosen to satisfy conditions:

$$(6.2) \quad \int_{-c+2d}^{\gamma_1(\xi)} \frac{(\lambda - \mu_1(\xi))(\lambda - \gamma_1(\xi))}{R_1^{1/2}(\lambda, \gamma_1(\xi))} d\lambda = 0, \quad 2c - 1 + \gamma_1(\xi) + 2\mu_1(\xi) = -2\xi.$$

The last one implies

$$\Phi_1(p, \xi) - g_1(p, \xi) = K_1(\xi) + \frac{k_1(\xi)}{\lambda} + O\left(\frac{1}{\lambda^2}\right), \quad \text{as } p = (\lambda, +) \rightarrow \infty_+,$$

where K_1 and k_1 are real valued constants. For the same reasons as above,

$$(6.3) \quad \frac{\partial K_1(\xi)}{\partial \xi} = \lim_{p \rightarrow \infty_+} \left(-\log(z_1(p)) + \int_1^p \tilde{\omega}_{\infty_+ \infty_-} \right) = \log \tilde{a} - \log d,$$

where $z_1(p)$ is defined by (2.7). The following lemma justifies the existence of this g -function, the proof is the same as for Lemma 3.1.

Lemma 6.1. *The functions $\gamma_1(\xi) \in (-1, 1)$ and $\mu_1(\xi) \in (-c + 2d, \gamma_1(\xi))$ satisfying (6.2) exist for $\xi \in (\xi_{cr,1}, \xi'_{cr,1})$. On this interval, $\gamma_1(\xi)$ is decreasing with $\gamma_1(\xi_{cr,1}) = 1$ and $\gamma_1(\xi'_1) = -1$.*

Next, denote

$$\int_{-c-2d}^{-c+2d} \frac{(\lambda - \mu_1)(\lambda - \gamma_1)}{R_1^{1/2}(\lambda + i0, \gamma_1)} d\lambda = iB_1(\xi), \quad \int_{\gamma_1}^1 \frac{(\lambda - \mu_1)(\lambda - \gamma_1)}{R_1^{1/2}(\lambda + i0, \gamma_1)} d\lambda = iB'_1(\xi).$$

For the same reasons as above (Lemma 3.3) we get

$$(6.4) \quad B_1(\xi) + B'_1(\xi) = \pi\xi,$$

respectively, that the function $e^{2g_1(p, \xi)t}$ does not have a jump on the intervals of $\mathbb{M}_1(\xi)$ with projection on $(-\infty, -c - 2d)$. Formula (6.4) also allows to represent the jump of g_1 along the contour $I_6 \cup I_6^*$, $I_6(\xi) = [-c + 2d, \gamma_1(\xi)]$, with the same orientation as the \mathbf{a} cycle in Section 5 as

$$g_{1,+}(p, \xi) - g_{1,-}(p, \xi) = -2itB_1(\xi) + 2\pi i\xi, \quad p \in I_6 \cup I_6^*.$$

The signature table for $\text{Re } g_1$ is depicted in Fig. 10. We introduce the contours Σ_1

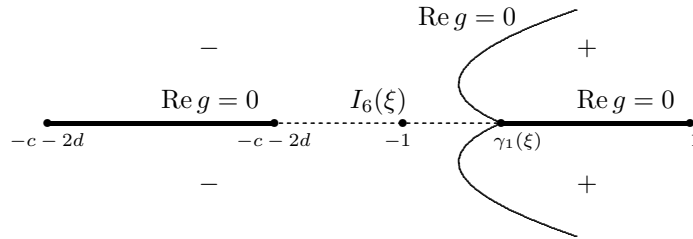


FIGURE 10. Sign of $\text{Re } g_1(p)$ on $\Pi_{1,U}(\xi)$

and $\Sigma_2(\xi)$ on $\mathbb{M}_1(\xi)$ along the boundaries of its sheets, with clockwise orientation, and consider the “left scattering data” problem I, (2.22)–III. Using the evident

symmetry of this problem to the problem studied above, we can perform successively all conjugation-deformation steps described in Section 4. Thus for Step 1, we multiply the solution $m(p) := m_1(p)$ by the diagonal matrix $D(p)$ with entries $d(p) = e^{t(\Phi_1(p) - g_1(p))}$, in Step 2 we transfer the resulting RH problem from \mathbb{M} to $\mathbb{M}_1(\xi)$ with complementary cuts along I_6 and I_6^* oriented as the \mathfrak{a} cycle above. In Step 3 we use the upper-lower factorization for the jump matrix on Σ_1 , then we factorize the matrix on $\Sigma_2(\xi)$ with the help of a function $F_1(p)$ which solves the following scalar problem: find a holomorphic function $F_1(p)$ on $\mathbb{M}_1(\xi) \setminus (\Sigma_2(\xi) \cup I_6 \cup I_6^*)$ with $F_1(p^*) = F_1^{-1}(p)$, bounded at ∞_{\pm} , and satisfying the jump conditions

$$(6.5) \quad F_{1,+}(p) = F_{1,-}(p) \begin{cases} |\chi(p)|, & p \in \Sigma_2(\xi), \\ e^{i\Delta_1(\xi)}, & p \in I_6 \cup I_6^*. \end{cases}$$

Here

$$(6.6) \quad \Delta_1(\xi) := \frac{i}{\Gamma_1} \int_{\Sigma_2(\xi)} \frac{\log |\chi(\lambda)| d\lambda}{R_1^{1/2}(\lambda, \gamma_1)}, \quad \Gamma_1 = 2 \int_{I_6} \frac{d\lambda}{R_1^{1/2}(\lambda, \gamma_1)}.$$

Recall that for large p , the values of the solution of the initial problem differ from the values of the solution obtained after Step 4, namely $m^4(p)$, by

$$(6.7) \quad m(p) = m^4(p) \begin{pmatrix} e^{t(\Phi_1(p) - g_1(p))} & 0 \\ 0 & e^{-t(\Phi_1(p) - g_1(p))} \end{pmatrix} \begin{pmatrix} F(p) & 0 \\ 0 & F^{-1}(p) \end{pmatrix}.$$

After Step 4, we arrive at the following model problem: Find a holomorphic vector-function (4.10) in the domain $\mathbb{M}_1(\xi) \setminus (\Sigma_2(\xi) \cup I_6(\xi) \cup I_6^*(\xi))$, bounded at ∞_+ , ∞_- , and satisfying the jump condition (4.11) with the jump matrix

$$(6.8) \quad v^{mod}(p) = \begin{cases} \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}, & p \in \Sigma_{2,u}(\xi), \\ \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, & p \in \Sigma_{2,\ell}(\xi), \\ \begin{pmatrix} e^{-2itB_1(\xi) + i\Delta_1(\xi)} & 0 \\ 0 & e^{2itB_1(\xi) - i\Delta_1(\xi)} \end{pmatrix}, & p \in I_6(\xi) \cup I_6^*(\xi), \end{cases}$$

the symmetry condition (4.12) and the normalization condition (4.13). The solution of this model problem can be derived similarly to the one in Section 5, and we describe it briefly.

Introduce the canonical basis \mathfrak{a}_1 and \mathfrak{b}_1 on the Riemann surface $\mathbb{M}_1(\xi)$ with \mathfrak{b}_1 cycle surrounding $[-c - 2d, -c + 2d]$ counterclockwise on the upper sheet and \mathfrak{a}_1 coinciding with $I_6 \cup I_6^*$. Let $\tilde{\omega}_{pp^*}$ be the normalized Abel differential of the third kind with poles at p and p^* , ζ is the holomorphic normalized Abel differential. Then the function

$$F_1(p) := \exp \left(\frac{1}{2\pi i} \int_{\Sigma_2(\xi)} \log |\chi| \tilde{\omega}_{pp^*} \right)$$

solves the conjugation problem (6.5). Denote by $\tau_1 = \int_{\mathfrak{b}_1} \tilde{\zeta}$ and let $\theta_1(v) = \theta(v | \tau_1)$. Set $A_1(p) = \int_{-c-2d}^p \tilde{\zeta}$ and denote by Ξ_1 the Riemann constant. Then $\theta_1(A_1(p) +$

$\frac{\tau_1}{2} - \frac{1}{2} - \Xi_1$) has a zero at γ_1 . Denote

$$(6.9) \quad \alpha_1(p) := \alpha_1(p, \xi) = \frac{\theta_1(A_1(p^*) + \frac{\tau_1(\xi)}{2} - \frac{1}{2} - \Xi_1(\xi) + \frac{tB_1(\xi)}{\pi} - \frac{\Delta_1(\xi)}{2\pi})}{\theta_1(A_1(p^*) + \frac{\tau_1(\xi)}{2} - \frac{1}{2} - \Xi_1(\xi))}$$

and $\delta_1(p) = \sqrt[4]{\frac{\pi(p) - \gamma_1}{\pi(p) - 1}}$. The vector

$$m^{mod}(p) = \frac{1}{\sqrt{\alpha_1(\infty_+) \alpha_1(\infty_-)}} (\delta_1(p) \alpha_1(p), \delta_1(p^*) \alpha_1(p^*))$$

solves the model problem (4.11)–(4.13) with jump matrix (6.8). Let U_1 be the \mathfrak{b} -period of the Abel differential of the second kind on $\mathbb{M}_1(\xi)$ with second order poles at ∞_+ and ∞_- and λ_1 be the \mathfrak{b} -period of $\tilde{\omega}_{\infty_+ \infty_-}$. If we use

$$\frac{tB_1(\xi)}{\pi} = -t \frac{U_1}{2\pi i} - n \frac{\lambda_1}{2\pi i}$$

to replace B_1 in (6.9) and define

$$(6.10) \quad z_1(n, t) := A(\infty_+) - n \frac{\lambda_1}{2\pi i} - t \frac{U_1}{2\pi i} + \frac{\tau_1(\xi)}{2} - \frac{1}{2} - \frac{\Delta_1(\xi)}{2\pi} - \frac{\lambda_1}{2\pi i} - \Xi_1(\xi),$$

we obtain for the first component of $m^{mod}(p)$ as $p \rightarrow \infty_+$

$$m_1^{mod}(\infty_+) = \sqrt{\frac{\theta_1(A_1(\infty_+) + \frac{\tau_1(\xi)}{2} - \frac{1}{2} - \Xi_1(\xi))}{\theta_1(A_1(\infty_-) + \frac{\tau_1(\xi)}{2} - \frac{1}{2} - \Xi_1(\xi))}} \sqrt{\frac{\theta_1(z_1(n, t))}{\theta_1(z_1(n-1, t))}}.$$

Following through the respective transformations in (6.7), (2.14), (6.3), we get in analog with (5.25) and (5.26) that $a^2(n, t) = a_q^2(n, t, \xi) + o(1)$, where

$$(6.11) \quad a_q^2(n, t, \xi) = \tilde{a}^2 \frac{\theta_1(z_1(n+1, t)) \theta_1(z_1(n-1, t))}{\theta_1^2(z_1(n, t))}.$$

Decomposing the first component of the solution with respect to large p we get

$$m_1^{mod}(p) = m_1^{mod}(\infty_+) \left(1 - \frac{1}{\Gamma_1 \lambda} \frac{\partial}{\partial w} \log \theta_1(z_1(n, t) + w) \Big|_{w=0} + \frac{\eta_1}{\lambda} + O\left(\frac{1}{\lambda^2}\right) \right),$$

where

$$\eta_1 := \eta_1(\xi) = \frac{1}{\Gamma_1} \frac{\partial}{\partial w} \log \left(\theta_1(A_1(\infty_-) + \frac{\tau_1}{2} - \frac{1}{2} - \Xi + w) \right) \Big|_{w=0} + \frac{1 - \gamma_1}{4}.$$

Respectively,

$$-B_1(n+1, t) = \left(tk_1(\xi) + \eta_1(\xi) + Q_1(\xi) - \frac{1}{\Gamma_1} \frac{\partial}{\partial w} \log \theta_1(z_1(n, t) + w) \Big|_{w=0} \right) (1 + o(1)),$$

where $\frac{dk_1(\xi)}{d\xi} = \tilde{b}(\xi) + c$ (see (3.23) and (3.3), second line) and

$$Q_1(\xi) = \int_{\Sigma_2(\xi)} \frac{\log |\chi(\lambda)| q_1(\lambda, \xi) d\lambda}{2\pi i R_1^{1/2}(\lambda, \gamma_1)} \in \mathbb{R}.$$

Applying now the same arguments as for (5.25) we get $b(n, t) = b_q(n, t, \xi) + o(1)$, where

$$(6.12) \quad b_q(n, t, \xi) = \tilde{b} + \frac{1}{\Gamma_1} \frac{\partial}{\partial w} \log \left(\frac{\theta(z_1(n-1, t) + w)}{\theta(z_1(n, t) + w)} \right) \Big|_{w=0}.$$

Theorem 6.2. *Let*

1. $\{a(n, t), b(n, t)\}$ be the solution of the problem (1.1), (2.1)–(2.4) as $n, t \rightarrow \infty$ in the domain $\xi_{cr,1} t < n < \xi'_{cr,1} t$, where $\xi_{cr,1}$ and $\xi'_{cr,1}$ are defined by (3.6)–(3.9);
 2. $\xi \in (\xi_{cr,1}, \xi'_{cr,1})$ be a parameter; $\gamma_1(\xi) \in (-1, 1)$ be defined by Lemma 6.1; $\mathbb{M}_1(\xi)$ be the Riemann surface of the function (6.1);
 3. $\{a_q(n, t, \xi), b_q(n, t, \xi)\}$ be the finite band solution defined by (6.11), (6.12), (6.10).
- Then in a vicinity of any ray $n = \xi t$ the solution of the problem (1.1), (2.1)–(2.4) has the following asymptotical behavior as $t \rightarrow +\infty$

$$a^2(n, t) = a_q^2(n, t, \xi) + o(1), \quad b(n, t) = b_q(n, t, \xi) + o(1).$$

7. ASYMPTOTICS OF THE SOLUTION IN THE DOMAIN $\xi'_{cr,1} < \xi < \xi'_{cr,2}$

In this region we work on the initial Riemann surface \mathbb{M} corresponding to the function (2.5). The g -functions associated with left and right RH problems (2.21) and (2.22) coincide up to the sign. Namely, introduce two points $-c + 2d < \mu_1(\xi) < \mu_2(\xi) < -1$ such that

$$\int_{-c+2d}^{-1} \frac{(\lambda - \mu_1(\xi))(\lambda - \mu_2(\xi))}{R^{1/2}(\lambda)} d\lambda = 0$$

and $\mu_1(\xi) + \mu_2(\xi) + c = -\xi$. The last equality implies that

$$g_{gap}(p, \xi) = \int_1^p \frac{(\zeta - \mu_1(\xi))(\zeta - \mu_2(\xi))}{R^{1/2}(\zeta)} d\zeta$$

has the following asymptotical behavior

$$g_{gap}(p) = \Phi_2(p) + O(1) = -\Phi_1(p) + O(1) \quad \text{as } p \rightarrow \infty_{\pm}.$$

Moreover, μ_1 and μ_2 exist as $\xi \in (\xi'_{cr,2}, \xi'_{cr,1})$ and the line $\operatorname{Re} g_{gap}(p) = 0$ crosses the real axis between these two points. We introduce the \mathfrak{b} and \mathfrak{a} cycles and all normalized Abel differentials on \mathbb{M} analogously to those on $\mathbb{M}(\xi)$, $\mathbb{M}_1(\xi)$. It is evident that both RH problems can be solved by an analogous procedure as above, but with Step 2 excluded. Thus, we get as asymptotics of the solution of the problem (1.1)–(2.4) two finite band solutions, described by (5.31)–(5.30) and (6.11)–(6.12) with the same theta function ($\tau = \tau_1$), but with different arguments: (5.20) and (6.10). Since $\underline{z}(n, t)$ and $\underline{z}_1(n, t)$ are defined on the same surface, then $\lambda = \lambda_1$, $U = U_1$, $\Xi = \Xi_1$, $A(\infty_+) = A_1(\infty_+)$, moreover, nothing depends on ξ . Thus,

$$\underline{z}_1(n, t) = \underline{z}(n, t) - \frac{\Delta}{2\pi} + \frac{1}{2} + \frac{\Delta_1}{2\pi},$$

where

$$\Delta := i \int_{\Sigma_1} \log |\chi| \zeta, \quad \Delta_1 := i \int_{\Sigma_2} \log |\chi| \zeta,$$

ζ is the holomorphic normalized Abel differential on \mathbb{M} , and Σ_1 and Σ_2 are defined in the beginning of Section 2. Hence, to prove that the asymptotics of the Toda lattice solution are the same from the right and from the left RH problems, it is sufficient to prove that $\Delta + \Delta_1 = \pi \pmod{2\pi}$ or

$$(7.1) \quad e^{i(\Delta + \Delta_1)} = -1.$$

To prove this formula recall that (cf. Lemma 5.2) the function

$$F(p) := \exp \left(\frac{1}{2\pi i} \int_{\Sigma} \log |\chi| \omega_{pp^*} \right), \quad \Sigma := \Sigma_1 \cup \Sigma_2,$$

is the unique solution of the following conjugation problem: to find a holomorphic function on $\mathbb{M} \setminus (\Sigma \cup \tilde{I} \cup \tilde{I}^*)$, where $\pi(\tilde{I}) = (-c + 2d, -1)$, bounded at ∞_{\pm} , and such that

$$(7.2) \quad F(p^*) = F^{-1}(p), \quad p \in \mathbb{M},$$

$$(7.3) \quad F_+(p) = F_-(p)|\chi(p)|, \quad p \in \Sigma.$$

Note that on the set $\tilde{I} \cup \tilde{I}^*$ this function has a jump,

$$(7.4) \quad F_+(p) = F_-(p)e^{i\tilde{\Delta}}, \quad p \in \tilde{I} \cup \tilde{I}^*, \quad \text{with } \tilde{\Delta} := i \int_{\Sigma} \log |\chi| \zeta = \Delta + \Delta_1.$$

The orientation on $\tilde{I} \cup \tilde{I}^*$ is the same as for the \mathbf{a} cycle. Note that the jump along $\tilde{I} \cup \tilde{I}^*$ can not be arbitrary. In fact, one can solve (7.3) without the Sokhotski–Plemelj formula as follows. Consider the function

$$\tilde{\delta}(\lambda) = \sqrt[4]{\frac{(\lambda + c)^2 - 4d^2}{\lambda^2 - 1}}, \quad \tilde{\delta}(2) > 0, \quad \lambda \in \mathbb{C} \setminus [-c - 2d, 1].$$

Let the interval $[-c - 2d, 1]$ be oriented in positive direction. Then

$$\tilde{\delta}_+(\lambda) = \tilde{\delta}_-(\lambda) \begin{cases} -i, & \lambda \in (-1, 1) \\ -1, & \lambda \in (-c + 2d, -1) \\ i, & \lambda \in (-c - 2d, -c + 2d). \end{cases}$$

Set $\delta(p) = \tilde{\delta}(\lambda)$ as $p = (\lambda, +) \in \Pi_U$ and $\delta(p) = \delta^{-1}(p^*)$. Then $\delta(p)$ solves the following conjugation problem

$$\delta_+(p) = \delta_-(p) \begin{cases} \left| \sqrt{\frac{(\pi(p)+c)^2 - 4d^2}{\pi(p)^2 - 1}} \right|, & p \in \Sigma, \\ -1, & p \in \tilde{I} \cup \tilde{I}^*. \end{cases}$$

On the other hand, the transmission coefficient $f(p) = T_2(p)$ is a single valued function on the upper sheet of \mathbb{M} and takes complex conjugated values in symmetric points of Σ . Continue f on the lower sheet by $f(p) = T_2^{-1}(p^*)$, $p = (\lambda, -)$. Then $f(p^*) = f^{-1}(p)$, $p \in \mathbb{M}$, and it is a solution which is bounded at ∞_{\pm} of the problem

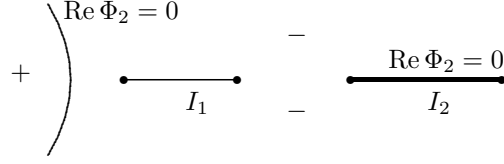
$$f_+(p) = f_-(p)|T_2(p)|^2, \quad p \in \Sigma.$$

We also observe that $f(p)$ and $\delta(p)$ are bounded at ∞_{\pm} . Taking into account (2.20) and (2.11) we conclude that the function $F(p) = \delta(p)f(p)$ solves the problem (7.2)–(7.3) and $F_+(p) = -F_-(p)$ as $p \in \tilde{I} \cup \tilde{I}^*$. Comparing this with (7.4) we get (7.1). Thus both RH problems provide the same finite band solution.

To formulate the result, recall that all objects introduced in Sections 5 and 6, namely, $A(p)$, τ , λ , U_0 , Δ , \tilde{a} , \tilde{b} do not depend on ξ for $\xi \in (\xi'_{cr,1}, \xi'_{cr,2})$. Respectively, the finite band solutions constructed in (5.31), (5.30) or (6.11), (6.12) do not depend on ξ .

Theorem 7.1. *Let $\{a_q(n, t), b_q(n, t)\}$ be finite band solutions constructed by (5.31), (5.30) on \mathbb{M} . In the domain $\xi'_{cr,1}t < n < \xi'_{cr,2}t$ the solution $\{a(n, t), b(n, t)\}$ of the problem (1.1), (2.1)–(2.4) is asymptotically close to $\{a_q(n, t), b_q(n, t)\}$ as $t \rightarrow \infty$ uniformly in $\frac{n}{t} \in [\xi'_{cr,1} + \varepsilon, \xi'_{cr,2} - \varepsilon]$.*

If $d = \frac{1}{2}$, so that I_1 and I_2 have equal length, then the solution of (1.1), (2.1)–(2.4) is close to the periodic Toda lattice solution as shown in [33].

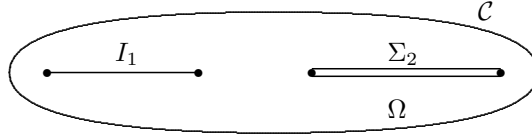
FIGURE 11. Signature table of $\text{Re } \Phi_2(p, \xi)$ for $\xi > \xi_{cr,2}$ on Π_U 8. ASYMPTOTICS OF THE SOLUTION IN THE DOMAINS $\xi > \xi_{cr,2}$ AND $\xi < \xi_{cr,1}$

We start with the domain $\xi > \xi_{cr,2}$, where we study the right RH problem I, (2.21)–III with $m(p) = m_2(p)$. The signature table of $\text{Re } \Phi(p)$ with $\Phi = \Phi_2$ is depicted in Fig. 11. Here Φ serves as the g -function itself and we omit Step 1. We apply Step 2 with $I_5 = I_1 = [-c - 2d, -c + 2d]$, which means that we switch from initial Riemann surface \mathbb{M} to the Riemann surface $\hat{\mathbb{M}}$ of the function $\sqrt{\lambda^2 - 1}$. Choosing I_1 with positive direction and I_1^* with negative direction and taking (3.2) into account, we have the equivalent conjugation problem $m_+(p) = m_-(p)v(p)$ with

$$(8.1) \quad v(p) = \begin{cases} \begin{pmatrix} 0 & -\overline{R(p)}e^{-2t\Phi(p)} \\ R(p)e^{2t\Phi(p)} & 1 \end{pmatrix}, & p \in \Sigma_2, \\ \begin{pmatrix} 1 & 0 \\ \chi(p)e^{2t\text{Re } \Phi(p)} & 1 \end{pmatrix}, & p \in I_1, \\ \begin{pmatrix} 1 & \chi(p)e^{-2t\text{Re } \Phi(p)} \\ 0 & 1 \end{pmatrix}, & p \in I_1^*, \end{cases}$$

satisfying (2.12) and initial normalization condition. As before we set $\chi(p) = -\chi(p^*)$ for $p \in I_1^*$ and orient I_1^* from $-c + 2d$ to $-c - 2d$.

Using (4.7) for Step 3 with $\Omega \subset \mathfrak{D} \cap \{p \in \Pi_U \mid \text{Re } \Phi(p) < 0\}$ as depicted in Fig. 12, and taking into account Lemma 4.1, we get that the jump matrix on

FIGURE 12. The lens contour for Step 3 on $\Pi_U(\xi)$.

$I_1 \cup I_1^*$ is simply the identity matrix. Thus on $\hat{\mathbb{M}}$ we get the following conjugation

problem: $m_+^3(p) = m_-^3(p)v^3(p)$ with

$$v^3(p) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & p \in \Sigma_2, \\ \begin{pmatrix} 1 & 0 \\ -R(p)e^{2t\Phi(p)} & 1 \end{pmatrix}, & p \in \mathcal{C}, \\ \begin{pmatrix} 1 & -R(p^*)e^{-2t\Phi(p)} \\ 0 & 1 \end{pmatrix}, & p \in \mathcal{C}^*. \end{cases}$$

These transformations did not affect the asymptotical behaviour of the initial solution. The jumps on $\mathcal{C} \cup \mathcal{C}^*$ are close to the identity matrix with a difference that is exponentially small with respect to t outside of small vicinities of $-1, 1$. Clearly the unique solution of the model problem $m_+^{mod}(p) = m_-^{mod}(p)$ on Σ_2 with standard normalization and symmetry conditions is the unit vector. Taking into account Lemma 2.2, we get $a(n, t) = \frac{1}{2} + o(1)$ and $b(n, t) = o(1)$ when $t \rightarrow \infty_+$ in the region $\frac{n}{t} > \xi_{cr,2}$.

Studying the left RH problem I, (2.22)–III in the domain $\frac{n}{t} < \xi_{cr,1}$, we get in the same manner $a(n, t) = d + o(1)$ and $b(n, t) = -c + o(1)$ when $t \rightarrow \infty_+$.

APPENDIX A. UNIQUENESS

Here we want to show that the Riemann–Hilbert problems from Theorem 2.3 have a unique solution. In fact since both problems for m_1 and m_2 can be easily transformed into each other by a simple conjugation it suffices to consider one of them. For convenience we show uniqueness for m_2 in the form of Section 8. To this end we introduce the meromorphic differential

$$d\Omega(p) = \frac{i d\lambda}{\pm \sqrt{\lambda^2 - 1}}, \quad p = (\lambda, \pm)$$

with simple poles at ∞_{\pm} . A brief inspection shows that $d\Omega$ is positive on Σ_2 and $i^{-1}d\Omega$ is positive in I_1 . We will also drop the index 2 for notational convenience and abbreviate

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It suffices to show that the associated vanishing problem, where the normalization condition III is replaced by the condition that the first component of $m(\infty_+)$ vanishes, has only the trivial solution. So let m be a solution of this vanishing problem and let \mathcal{C} be the closed contour from Fig. 12 oriented counterclockwise. Denote by m^\dagger the adjoint (transpose and complex conjugate) of a vector/matrix. Since there is no residue at ∞_+ we obtain

$$\begin{aligned} 0 &= \int_{\mathcal{C}} m(p)m^\dagger(\overline{p^*})d\Omega(p) \\ &= \int_{I_1} m_+(p)m_+^\dagger(p^*)d\Omega(p) - \int_{I_1} m_-(p)m_-^\dagger(p^*)d\Omega(p) + \int_{\Sigma_2} m_+(p)m_-^\dagger(p)d\Omega(p) \\ &= \int_{I_1} (m_+(p)\sigma_1 m_-^\dagger(p) - m_-(p)\sigma_1 m_+^\dagger(p))d\Omega(p) + \int_{\Sigma_2} m_+(p)m_-^\dagger(p)d\Omega(p). \end{aligned}$$

Now using $m(p) = (m_1(p), m_2(p))$, the jump conditions

$$(A.1) \quad m_{1,+} - m_{1,-} = \chi e^{2t \operatorname{Re} \Phi} m_{2,-}, \quad m_{2,+} = m_{2,-}, \quad \text{on } I_1,$$

$$(A.2) \quad m_{2,+} = m_{2,-} - \bar{R} e^{-2t \Phi} m_{1,-}, \quad m_{1,+} = R e^{2t \Phi} m_{2,-}, \quad \text{on } \Sigma_2,$$

together with $\chi(p) = i|\chi(p)|$, $p \in I_1$, and $\operatorname{Re} \Phi(p) = 0$, $p \in \Sigma_2$, imply

$$\begin{aligned} 0 = & 2 \int_{I_1} |\chi(p)| e^{2t \operatorname{Re} \Phi(p)} |m_{2,-}(p)|^2 d\Omega(p) + \int_{\Sigma_2} |m_{2,-}(p)|^2 d\Omega(p) \\ & + i2 \operatorname{Im} \int_{\Sigma_2} R(p) e^{2t \Phi(p)} m_{2,-}(p) \overline{m_{1,-}(p)} d\Omega(p). \end{aligned}$$

Since the first two integrals are positive and the last is purely imaginary this shows $m_{2,-}(p) = 0$ for $p \in I_1 \cup \Sigma_2$. By (A.1) $m_{2,+}(p) = m_{2,-}(p) = 0$ for $p \in I_1$ and so m_1 also has no jump along I_1 . In particular, m is holomorphic in a neighborhood of I_1 and consequently vanishes on the upper sheet. By symmetry it also vanishes on the lower sheet which finally shows $m(p) \equiv 0$ and establishes uniqueness.

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FACULTY OF MATHEMATICS, UNIVERSITY OF VIENNA, OSKAR-MORGENSTERN-PLATZ 1, 1090 WIEN, AUSTRIA, AND INSTITUTE FOR LOW TEMPERATURE PHYSICS, 47, LENIN AVE, 61103 KHARKIV, UKRAINE

E-mail address: iraegorova@gmail.com

FACULTY OF MATHEMATICS, UNIVERSITY OF VIENNA, OSKAR-MORGENSTERN-PLATZ 1, 1090 WIEN, AUSTRIA, AND INTERNATIONAL ERWIN SCHRÖDINGER INSTITUTE FOR MATHEMATICAL PHYSICS, BOLTZMANNGASSE 9, 1090 WIEN, AUSTRIA

E-mail address: Johanna.Michor@univie.ac.at

URL: <http://www.mat.univie.ac.at/~jmichor/>

FACULTY OF MATHEMATICS, UNIVERSITY OF VIENNA, OSKAR-MORGENSTERN-PLATZ 1, 1090 WIEN, AUSTRIA, AND INTERNATIONAL ERWIN SCHRÖDINGER INSTITUTE FOR MATHEMATICAL PHYSICS, BOLTZMANNGASSE 9, 1090 WIEN, AUSTRIA

E-mail address: Gerald.Teschl@univie.ac.at

URL: <http://www.mat.univie.ac.at/~gerald/>